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Output Feedback Consensus for Multiple Mobile Robots using Negative-Imaginary System Properties via Nonlinear State Feedback*

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Abstract— **This paper addresses the output consensus problem for networked mobile robots by using negative-imaginary (NI) systems theory. It is shown that there exists a nonlinear state feedback control law for each wheeled mobile robot (WMR) which results in satisfying the output strictly negative-imaginary (OSNI) property of the closed-loop system. Nonlinear NI controllers are placed in positive feedback with the rendered OSNI systems through a fixed network topology. It is shown that the nonlinear NI/OSNI properties are retained in the networked configuration and the output consensus result is proved. The proposed result is robust against NI uncertainties and is applicable to multi-robot systems without considering any linearization assumption. Simulation results are presented to validate the efficacy of the proposed control scheme.**

I. INTRODUCTION

Research in the area of cooperative control of multi-agent systems (MAS) has been increasingly active over the past decade. MAS may comprise of multiple unmanned aerial vehicles, autonomous underwater vehicles, or ground portable robots [1]. Studying cooperation among multiple mobile robots is crucial due to their wide-ranging real-world applications, which are motivated from diverse natural phenomena [2]–[5]. Such collaboration of multiple agents aim to achieve specific collective behaviors or to accomplish goals that are complex and time consuming for a single agent. Understandably, ensuring all the mobile robots autonomously work towards a cooperatively decided objective is demanding. Inherent uncertainties in the system and the environment impose further challenges. Therefore, it is essential to develop advanced control strategies and distributed robust control laws for a network of multiple robots to achieve the desired collective behavior.

Developing control strategies leveraging on the system properties pose beneficial due to already-established synthesis frameworks and inherent robustness results. In this paper, the consensus problem for multiple nonholonomic wheeled mobile robots (WMR) is solved by utilizing the system's negative-imaginary (NI) properties. This choice is motivated by the observation that, with state-feedback, the kinematics of a WMR can be rendered nonlinear NI, and hence, NI-domain stability results can provide straightforward yet effective cooperative control approaches without relying on linearization as in [6].

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In order to control flexible structures with collocated force actuators and position measurements, the NI systems were initially introduced in [7]. A linear time invariant (LTI) system $G(s)$ is said to have negative-imaginary frequency response when $j[G(j\omega)-G(j\omega)^*] \geq 0, \forall \omega \in (0,\infty)$ [7], [8]. The NI systems theory gained significant attention, mainly because of the robust stability result of NI systems with positive feedback interconnection under suitable assumptions [7], [9]. Recent advances on the LTI NI theories can be found in [10]–[15], only to mention a few. Extension of NI characterization and stability theories for nonlinear systems is actively being explored [16]–[18]. A principal application domain of the interconnected stability result of nonlinear NI systems is robust control of dynamical systems, for example, guaranteeing robustness against nonlinear NI spillover dynamics. Recently, researchers have found the applications of the NI systems theory to solve the MAS control problems [19], [20]. In [19], it is first shown that the consensus control problem for a network of homogeneous NI systems (i.e., NI systems with identical dynamics) can be cast as an internal stability problem. The result is then extended in [20] for heterogeneous NI agents (i.e., agents with non-identical NI dynamics). However, these results were applicable only for LTI NI dynamics. Robust consensus results for networked homogeneous and heterogeneous *nonlinear* NI systems are explored in [21] and [22], respectively.

Motivated by [22], this paper aims to tackle the output feedback consensus problem for WMRs by enforcing a strict NI property on individual agents via state feedback. We consider the actual kinematic model of each WMR and develop a distributed state-feedback making each system dynamics to satisfy output strictly negative-imaginary (OSNI) properties, a subclass of NI. We establish that the OSNI properties are maintained for the networked plant with a directional information flow characterized by a balanced and strongly connected graph. We then obtain conditions that ensure robust output feedback consensus by applying nonlinear networked NI controller in positive feedback. Simulation results are presented in order to illustrate the theoretical developments.

Notation: Let $\mathbb{C}^{m\times n}$ and $\mathbb{R}^{m\times n}$ represent the set of complex and real matrices of $m\times n$ dimension, respectively. I_n indicates the identity matrix of dimension $n\times n$ and $\mathbf{1}_n$ is the $n\times 1$ vector where all entries are 1. The transpose and the conjugate transpose of matrix P are denoted by $P^\top$, and P^*, respectively. $\mathcal{N}(P)$ represents the null space of the matrix P. $||\cdot||$ denotes the Euclidean norm of a vector. $A\otimes B$ represents the Kronecker product of matrices A and B.

II. PRELIMINARIES

A. Nonlinear negative-imaginary systems

Consider the nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \tag{1a}$$
$$y(t) = h(x(t)), \tag{1b}$$

where $f : \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^q$ is Lipschitz in x, and $h : \mathbb{R}^q \to \mathbb{R}^m$ is continuously differentiable.

Definition 1 ([22]): System (1) is considered as a nonlinear NI system when there exists a continuously differentiable positive semidefinite storage function $V : \mathbb{R}^q \to \mathbb{R}$ such that

$$\dot{V}(x(t)) \leq u(t)^\top \dot{y}(t), \quad \forall t \geq 0. \tag{2}$$

Definition 2 ([22]): System (1) is considered as a nonlinear OSNI system when there exist a continuously differentiable positive semidefinite storage function $V : \mathbb{R}^q \to \mathbb{R}$ and a constant $\delta > 0$ such that

$$\dot{V}(x(t)) \leq u(t)^\top \dot{y}(t) - \delta \|\dot{y}(t)\|^2, \quad \forall t \geq 0. \tag{3}$$

Here, system (1) is called OSNI with an output strictness δ. Clearly, OSNI systems form a subclass of NI systems.

B. Graph theory preliminaries

One can mathematically represent a graph $\mathcal{G}$ by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{v_1, \ldots, v_n\}$ denotes a nonempty finite set of n nodes/vertices and $\mathcal{E} = \{e_1, \ldots, e_l\} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of ordered pair of vertices called edges, which are to be interpreted as enabling information flow between the vertices of the corresponding edge. If the flow of information is direction sensitive, then $\mathcal{G}$ is a directed graph. A directed graph is strongly connected when there exist directed paths between any pair of vertices in both directions. In a balanced directed graph $\mathcal{G}$, for each vertex v, the number of edges directed towards v (in-degree) is equal to the number of edges directed away from v (out-degree). In this work, we consider a balanced and strongly connected graph to model the communication links among the mobile robots. The incidence matrix $Q = [q_{ve}] \in \mathbb{R}^{n \times l}$ for a directed graph is described as,

$$Q := \begin{cases} q_{ve} = 1 & \text{when } v \text{ is the parent node of edge } e, \\ q_{ve} = -1 & \text{when } v \text{ is the child node of edge } e, \\ q_{ve} = 0 & \text{when } v \text{ is not connected to edge } e. \end{cases}$$

For a balanced and strongly connected graph, Q has a unique zero eigenvalue associated with the eigenvector $\mathbf{1}_n$, and all other eigenvalues being real and positive. Therefore, $\mathcal{N}(Q) = \text{span}\{\mathbf{1}_n\}$.

C. Multi-robot model

In the literature [23]–[25], the dynamics of multiple wheeled mobile robots is explored through the particle model, where each mobile robot moves on the horizontal plane. A set of $n \geq 2$ WMRs are considered here and the equations of motion of the i-th robot is given as follows:

$$\dot{r}_i = u_{i1} e^{j\phi_i}, \tag{4a}$$
$$\dot{\phi}_i = u_{i2}, \qquad \forall i \in \{1, \cdots, n\}, \tag{4b}$$

where $r_i \in \mathbb{C}$ denotes the position, ϕ_i represents the heading angle of the i-th robot, and $j = \sqrt{-1}$. Here, u_{i1} and u_{i2} are the linear and angular velocities of the i-th robot, respectively. The describing equations of (4) can be simply interpreted from Fig. 1. The same model can also be

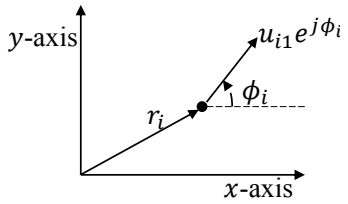


Fig. 1: Position and orientation of the i-th robot

represented in a planar ($\mathbb{R}^2$) structure. We, therefore, describe the kinematic model of the i-th robot as:

$$\dot{x}_i = u_{i1} \cos(\phi_i), \tag{5a}$$
$$\dot{y}_i = u_{i1} \sin(\phi_i), \quad \forall i \in \{1, \cdots, n\} \tag{5b}$$
$$\dot{\phi}_i = u_{i2}, \tag{5c}$$

where $\begin{bmatrix} x_i & y_i \end{bmatrix}^\top \in \mathbb{R}^2$ denotes the position and $\phi_i \in \mathbb{R}$ is the orientation of the i-th robot. Let $U_i = \begin{bmatrix} u_{i1} & u_{i2} \end{bmatrix}^\top \in \mathbb{R}^2$, where u_{i1} and u_{i2} are the linear and angular velocities of the i-th robot, respectively, which are given as inputs to the i-th robot. Here, $Y_i = \begin{bmatrix} x_i & y_i \end{bmatrix}^\top \in \mathbb{R}^2$ is considered as the output of the i-th robot.

III. ENFORCING NONLINEAR OSNI PROPERTY VIA STATE FEEDBACK

We consider WMRs with the kinematics satisfying (5). In this section, it is shown that there exists nonlinear state feedback control laws for each nonholonomic WMR which result in satisfying the OSNI property of the closed-loop. This can be termed as the state feedback equivalence to a nonlinear OSNI system. The state feedback equivalence to *LTI* NI system was defined in [26]; extension of that definition for *nonlinear* NI (OSNI) systems is provided in the following.

Definition 3: A system (1) is considered as state feedback equivalent to an NI (OSNI) system when there exists a nonlinear state feedback control law

$$u = k_c(x, w),$$

with $k_c : \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^m$, such that the resulting closed-loop system with the new input $w \in \mathbb{R}^m$ is nonlinear NI (OSNI).

The following lemma shows that each WMR of the form (5) is state feedback equivalent to an OSNI system.

Lemma 1: Let H_p be a WMR system of the form (5). There exists a nonlinear state feedback control

$$U_i = k_c(x_i, y_i, \phi_i, W_i) \tag{6}$$

such that the transformed system (H_{pcl}) from the new input $W_i \in \mathbb{R}^2$ to the output Y_i (as shown in Fig. 2) becomes OSNI.

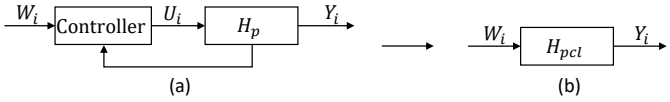


Fig. 2: The i-th robotic system model H_p is transformed to an OSNI system, H_{pcl}, via nonlinear state feedback

Proof: Let us construct a possible storage function $V_i(x_i, y_i, \phi_i)$ for the i-th robot as

$$V_i(x_i, y_i, \phi_i) = \frac{1}{2}x_i^2 + \frac{1}{2}y_i^2. \tag{7}$$

Clearly, $V_i(x_i, y_i, \phi_i)$ is positive semidefinite. Consider the control law

$$\begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix} = \begin{bmatrix} \cos(\phi_i) & \sin(\phi_i) \\ -\sin(\phi_i) & \cos(\phi_i) \end{bmatrix} \left(\begin{bmatrix} w_{i1} \\ w_{i2} \end{bmatrix} - \begin{bmatrix} x_i \\ y_i \end{bmatrix} \right), \tag{8}$$

where $W_i = \begin{bmatrix} w_{i1} & w_{i2} \end{bmatrix}^\top \in \mathbb{R}^2$ is the new input to the system. The resulting system H_{pcl} therefore becomes

$$\dot{x}_i = [\cos(\phi_i)(w_{i1} - x_i) + \sin(\phi_i)(w_{i2} - y_i)]\cos(\phi_i), \tag{9a}$$

$$\dot{y}_i = [\cos(\phi_i)(w_{i1} - x_i) + \sin(\phi_i)(w_{i2} - y_i)]\sin(\phi_i), \tag{9b}$$

$$\dot{\phi}_i = [-\sin(\phi_i)(w_{i1} - x_i) + \cos(\phi_i)(w_{i2} - y_i)], \tag{9c}$$

$$Y_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} w_{i1} \\ w_{i2} \end{bmatrix} - \begin{bmatrix} \cos(\phi_i) & -\sin(\phi_i) \\ \sin(\phi_i) & \cos(\phi_i) \end{bmatrix} \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix}. \tag{9d}$$

Now, differentiating (7) with respect to time along the trajectories of (9), we obtain

$$\begin{aligned} \dot{V}_i &= x_i\dot{x}_i + y_i\dot{y}_i \\ &= [w_{i1} - \cos(\phi_i)u_{i1} + \sin(\phi_i)u_{i2}]\,\dot{x}_i \\ &\quad + [w_{i2} - \sin(\phi_i)u_{i1} - \cos(\phi_i)u_{i2}]\,\dot{y}_i \\ &= \begin{bmatrix} w_{i1} & w_{i2} \end{bmatrix} \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix} - \begin{bmatrix} \dot{x}_i & \dot{y}_i \end{bmatrix} \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix} \\ &= W_i^\top \dot{Y}_i - \|\dot{Y}_i\|^2. \end{aligned} \tag{10}$$

Here, (10) satisfies Definition 2. Therefore, the system (9) is nonlinear OSNI. ■

IV. OUTPUT FEEDBACK CONSENSUS FOR THE NETWORK OF WMRS

This section establishes the output feedback consensus result for a network of WMRs using networked nonlinear NI controllers. In this work, the robust output feedback consensus is achieved by constructing a control framework with positive feedback interconnection as shown in Fig. 4. The communication links between the network of robots are modeled using a balanced and strongly connected graph $\mathcal{G}$, having a corresponding incidence matrix Q. The nodes correspond to the plants and the controllers work on the difference between the outputs of a pair of plants communicating through an edge. To derive the consensus result, we first show that the NI property is retained for the networked nonlinear NI controllers, and the nonlinear OSNI property is maintained for the networked WMRs with state feedback. The following result will be applicable to homogeneous as well as heterogeneous multi-robot systems.

A. NI property for a network of NI controllers

Consider a series of $l \geq 2$ possibly nonlinear NI controllers H_c of the following form

$$\dot{x}_{ck} = f_c(x_{ck}, u_{ck}), \tag{11a}$$

$$y_{ck} = h_c(x_{ck}), \qquad \forall k \in \{1, \cdots, l\}, \tag{11b}$$

where $f_c : \mathbb{R}^p \times \mathbb{R}^2 \to \mathbb{R}^p$ is Lipschitz in x_{ck}, $h : \mathbb{R}^p \to \mathbb{R}^2$ is a continuously differentiable function, and the subscript "c" is used to indicate *controller*. The index k will be used to specify the k-th NI controller in the network.

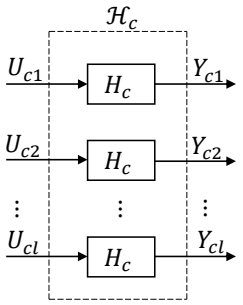


Fig. 3: Networked controller $\mathcal{H}_c$ consisting of l nonlinear NI controllers H_c

The networked NI controller $\mathcal{H}_c$ under consideration is depicted in Fig. 3, and the state-space model of the networked controller $\mathcal{H}_c$ is represented as

$$\dot{X}_c = \begin{bmatrix} f(x_{c1}, u_{c1}) \\ f(x_{c2}, u_{c2}) \\ \vdots \\ f(x_{cl}, u_{cl}) \end{bmatrix}, \quad Y_c = \begin{bmatrix} h(x_{c1}) \\ h(x_{c2}) \\ \vdots \\ h(x_{cl}) \end{bmatrix}, \tag{12}$$

where $X_c = \begin{bmatrix} x_{c1}^\top, \cdots, x_{cl}^\top \end{bmatrix}^\top \in \mathbb{R}^{lp}$, $U_c = \begin{bmatrix} u_{c1}^\top, \cdots, u_{cl}^\top \end{bmatrix}^\top \in \mathbb{R}^{2l}$, and $Y_c = \begin{bmatrix} y_{c1}^\top, \cdots, y_{cl}^\top \end{bmatrix}^\top \in \mathbb{R}^{2l}$.

The following lemma shows that nonlinear NI property is retained in the networked NI controllers.

Lemma 2: If a system H_c is nonlinear NI, the networked system $\mathcal{H}_c = \text{diag}(H_c, \cdots, H_c)$, as shown in Fig. 3 is also nonlinear NI.

Proof: For an NI system H_c, there exists a positive definite storage function $V_k(x_{ck})$ such that $\dot{V}_k(x_{ck}) \leq u_{ck}^\top \dot{y}_{ck}$, $\forall k \in \{1, 2, \cdots, l\}$. For the networked system $\mathcal{H}_c$, the storage function is defined as $\hat{V}_2 = \sum_{k=1}^{l} V_k(x_{ck}) > 0$. The time derivative of $\hat{V}_2$ is

$$\dot{\hat{V}}_2 = \sum_{k=1}^{l} \dot{V}_k(x_{ck}) \leq \sum_{k=1}^{l} u_{ck}^\top \dot{y}_{ck} = U_c^\top \dot{Y}_c. \tag{13}$$

Therefore, the networked controller $\mathcal{H}_c$ is NI. ■

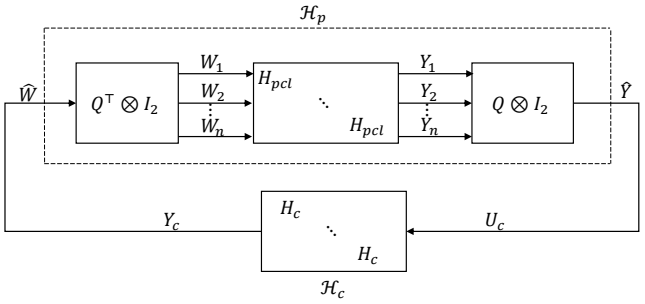


Fig. 4: Positive feedback interconnection of networked OSNI systems and networked NI controllers via directed graph topology

B. OSNI property for a network of WMRs

In this subsection, we show that the OSNI property is preserved for networked WMRs.

A set of $n \geq 2$ WMRs H_p, as given in (5), are considered. Nonlinear state feedback control law (8) for each WMR makes individual closed-loop system satisfy nonlinear OSNI property. The communication connections between the networked nonlinear OSNI plants H_{pcl} is modeled using a balanced and strongly connected directed graph $\mathcal{G}$ associated with the incidence matrix Q. The networked plant $\mathcal{H}_p$ consists of n closed-loop nominal plants H_{pcl} with input $W_i \in \mathbb{R}^2$ and output $Y_i \in \mathbb{R}^2$, $\forall i \in \{1, \cdots, n\}$, as illustrated in Fig. 4. Let $W = \begin{bmatrix} W_1^\top & W_2^\top & \cdots & W_n^\top \end{bmatrix}^\top \in \mathbb{R}^{2n}$ and $Y = \begin{bmatrix} Y_1^\top & Y_2^\top & \cdots & Y_n^\top \end{bmatrix}^\top \in \mathbb{R}^{2n}$ represent the input and output of the system $\text{diag}\,(H_{pcl}, \cdots, H_{pcl})$. As shown in Fig. 4, the networked plant $\mathcal{H}_p$ is connected in positive feedback with the networked NI controller $\mathcal{H}_c$ to ensure consensus. The difference in the outputs of any pair of systems H_{pcl} that are connected by an edge is fed as the input to each controller H_c. The system $\mathcal{H}_p$ takes the outputs of the controller $\mathcal{H}_c$ as input $\hat{W}$ and distribute to each system H_{pcl} through the network topology.

Based on the configuration shown in Fig. 4, the input to the system $\mathcal{H}_c$ is

$$U_c = (Q \otimes I_2)Y, \tag{14}$$

and W can be expressed as

$$W = (Q^\top \otimes I_2)Y_c. \tag{15}$$

Here, Y_c is the output of $\mathcal{H}_c$ as introduced in (12).

Now, we are ready to prove that the overall networked plant remains OSNI.

Lemma 3: The networked nonlinear plant $\mathcal{H}_p$, which consists of n nonlinear OSNI systems (H_{pcl}) interconnected through a balanced and strongly connected graph with corresponding incidence matrix Q, as shown in Fig. 4, is also a nonlinear OSNI.

Proof: For an OSNI system H_{pcl}, we consider $V_i(x_i, y_i, \phi_i)$ as a positive semidefinite storage function such that $\dot{V}_i(x_i, y_i, \phi_i) = W_i^\top \dot{Y}_i - \|\dot{Y}_i\|^2$, $\forall i \in \{1, \cdots, n\}$. For the networked system $\mathcal{H}_p$, the storage function can be constructed as $\hat{V}_1 = \sum_{i=1}^{n} V_i(x_i, y_i, \phi_i) \geq 0$. Time derivative along the system trajectories yields

$$\begin{aligned} \dot{\hat{V}}_1 &= \sum_{i=1}^{n} \dot{V}_i(x_i, y_i, \phi_i) \\ &= \sum_{i=1}^{n} \left(W_i^\top \dot{Y}_i - \|\dot{Y}_i\|^2 \right) \quad \text{[From (10)]} \\ &= W^\top \dot{Y} - \|\dot{Y}\|^2. \end{aligned} \tag{16}$$

Since

$$W = (Q^\top \otimes I_2)\hat{W}, \quad \text{and} \quad \hat{Y} = (Q \otimes I_2)Y, \tag{17}$$

from the block diagram of Fig. 4, we can write

$$W^\top Y = [(Q^\top \otimes I_2)\hat{W}]^\top Y = \hat{W}^\top \hat{Y}. \tag{18}$$

Utilizing (16) and (18), one can claim that $\exists \delta > 0$ such that

$$\begin{aligned} \dot{\hat{V}}_1 &= \hat{W}^\top \dot{\hat{Y}} - \|\dot{Y}\|^2 \\ &\leq \hat{W}^\top \dot{\hat{Y}} - \delta \|(Q \otimes I_2)\dot{Y}\|^2 \\ &= \hat{W}^\top \dot{\hat{Y}} - \delta \|\dot{\hat{Y}}\|^2. \end{aligned} \tag{19}$$

Hence, the system $\mathcal{H}_p$ is nonlinear OSNI with output strictness δ. ■

The consensus result for networked WMRs using networked nonlinear NI controllers is proved in the following subsection.

C. Output feedback consensus of a networked WMRs

We now define output feedback consensus for the networked system $\mathcal{H}_p$, which consists of n WMRs.

Definition 4: A distributed output feedback control law is said to achieve the output feedback consensus for a network of n systems of the form (5) with i-th output Y_i if $\|Y_i(t) - Y_j(t)\| \to 0$ as $t \to +\infty$, for all $i, j \in \{1, 2, \ldots, n\}$.

Next, we present the consensus result of the networked WMRs H_{pcl} in $\mathcal{H}_p$. In this work, the consensus problem is solved considering only the position of the robots while ignoring the orientations. Nevertheless, boundedness of the robots' orientation would be ensured.

For this consensus result, we assume:

Assumption I: In the open-loop interconnection depicted in Fig. 5, a constant input $\overline{U}_c$ is given to the system $\mathcal{H}_c$, we get a corresponding output Y_c . The input of the system represented by $\mathcal{H}_p$ is $\hat{W} \equiv Y_c$. If the system $\mathcal{H}_p$ yields a constant output, $\hat{Y}(t) \equiv \overline{\hat{Y}}$, then there exists a $\beta \in (0, 1)$ such that

$$\overline{U}_c^\top \overline{\hat{Y}} \leq \beta \|\overline{U}_c\|^2. \tag{20}$$

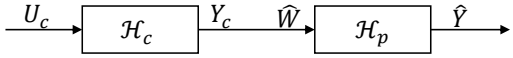


Fig. 5: Open-loop interconnection of networked NI system $\mathcal{H}_c$ and networked OSNI system $\mathcal{H}_p$

Theorem 1: A network of wheeled mobile robots H_{pcl} of the form (9), modeled by a balanced and strongly connected directed graph $\mathcal{G}$ associated with the incidence matrix Q, is considered. Also, controller H_c for each edge of the graph is considered to be NI of the form (11). Suppose Assumption I is satisfied by the open-loop connection of $\mathcal{H}_p$ and $\mathcal{H}_c$, and let a combined storage function be defined as

$$\mathcal{S} = \hat{V}_1 + \hat{V}_2 - \hat{Y}^\top Y_c, \tag{21}$$

where the storage functions $\hat{V}_1$ and $\hat{V}_2$, associated with the systems $\mathcal{H}_p$ and $\mathcal{H}_c$ fulfilling (19) and (13), are positive semidefinite and positive definite, respectively. The output of $\mathcal{H}_p$ being $\hat{Y}$ and the output of $\mathcal{H}_c$ being Y_c, the output feedback consensus can be established utilizing the control laws (14), (15), and (12), as shown in Fig. 4.

Proof: Differentiating the combined storage function of the overall closed-loop with respect to time along the system trajectories, we get

$$\begin{aligned}\dot{\mathcal{S}} &= \dot{\hat{V}}_1 + \dot{\hat{V}}_2 - \dot{\hat{Y}}^\top Y_c - \hat{Y}^\top \dot{Y}_c \\ &\leq \hat{W}^\top \dot{\hat{Y}} - \delta \|\dot{\hat{Y}}\|^2 + U_c^\top \dot{Y}_c - \dot{\hat{Y}}^\top \hat{W} - U_c^\top \dot{Y}_c \\ &= -\delta \|\dot{\hat{Y}}\|^2 \leq 0. \end{aligned} \tag{22}$$

This confirms the Lyapunov stability of the overall system. According to (22), $\dot{\mathcal{S}}$ can be zero only when $\dot{\hat{Y}} = 0$. Therefore, $\dot{\hat{Y}} = 0 \implies \hat{Y} \equiv \overline{\hat{Y}}$. Since $\overline{\hat{Y}} = \overline{U}_c$, from (20), we obtain $\overline{U}_c^\top \overline{\hat{Y}} = \|\overline{U}_c\|^2 \leq \beta \|\overline{U}_c\|^2$, which can be satisfied only when $\overline{\hat{Y}} = \overline{U}_c = 0$. Thus, $\overline{\hat{Y}} = 0 \implies \|Y_i(t) - Y_j(t)\| = 0$, for all $i, j \in \{1, \ldots, n\}$, by using the property of the incidence matrix of a balanced and strongly connected directed graph. Therefore, the outputs of all WMRs will asymptotically reach to the same value. Consequently, the output feedback consensus for networked WMRs is proved. ■

Remark 1: The proposed cooperative control scheme guarantees robustness against certain NI domain uncertainties in the form of unmodelled or spillover dynamics in each agent of the network. This is due to the fact that the networked WMR systems ($\mathcal{H}_p$) is made to satisfy OSNI properties, and according to the nonlinear NI-OSNI stability result [22], the closed-loop interconnection is robust against NI uncertainties in the individual system's model.

V. ILLUSTRATIVE EXAMPLE

We consider a networked system consisting $n = 4$ robots with the initial positions and heading angles as $\begin{bmatrix} x_1(0) & \cdots & x_4(0) \end{bmatrix}^\top = \begin{bmatrix} -20 & -15 & -5 & 10 \end{bmatrix}^\top$, $\begin{bmatrix} y_1(0) & \cdots & y_4(0) \end{bmatrix}^\top = \begin{bmatrix} 40 & -20 & -30 & -10 \end{bmatrix}^\top$, and $\begin{bmatrix} \phi_1(0) & \cdots & \phi_4(0) \end{bmatrix}^\top = \begin{bmatrix} 0^\circ & 30^\circ & 45^\circ & 60^\circ \end{bmatrix}^\top$, connected by a balanced and strongly connected graph $\mathcal{G}$ as depicted in Fig. 6. The incidence matrix of the directed graph is

$$Q = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

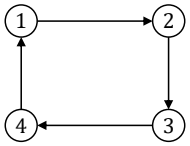


Fig. 6: A balanced and strongly connected graph depicting the communication links between the four mobile robots

Now, we apply a networked nonlinear NI controller for all the four edges. Let the NI controllers H_c have the following nonlinear dynamics:

$$\begin{aligned} \dot{x}_{ck1} &= a x_{ck2}, \\ \dot{x}_{ck2} &= -a x_{ck1} - x_{ck2}^3 + b(u_{ck1} + u_{ck2}), \\ y_{ck1} &= y_{ck2} = x_{ck1} \qquad \forall k \in \{1, 2, 3, 4\}, \end{aligned}$$

where $a, b > 0$. The controller H_c is NI with the positive definite storage function $V_k(x_{ck1}, x_{ck2}) = \frac{2}{ab}(x_{ck1}^2 + x_{ck2}^2)$, $\forall k \in \{1, 2, 3, 4\}$, which satisfy the conditions of the Lyapunov function for the complete networked system for $a = 2$ and $b = 1/8$. According to Theorem 1, output feedback consensus can be ensured using the control law $W = (Q^\top \otimes I_2) Y_c$, and $U_c = (Q \otimes I_2) Y$, as shown in Fig. 7, which depicts that the position of all WMRs $Y_i = \begin{bmatrix} x_i & y_i \end{bmatrix}^\top$, $(i = 1, 2, 3, 4)$ converge to the origin establishing consensus. Figs. 8 and 9 show that output-1 (x_i) and output-2 (y_i) of all robots converge to the same trajectories. The control inputs of all WMRs $U_i = \begin{bmatrix} u_{i1} & u_{i2} \end{bmatrix}^\top$, $(i = 1, 2, 3, 4)$ that achieve consensus, are shown in Figs. 10 and 11.

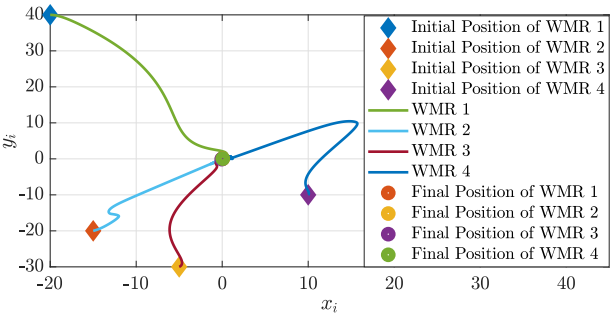


Fig. 7: Output consensus for the networked WMRs

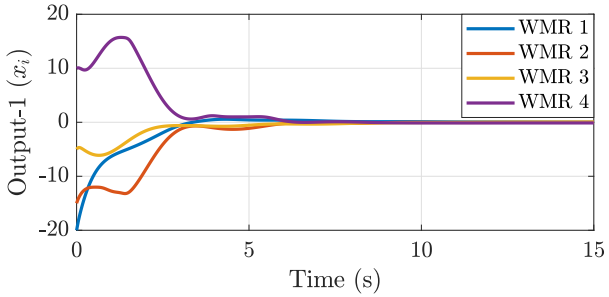


Fig. 8: Output-1 of all four WMRs

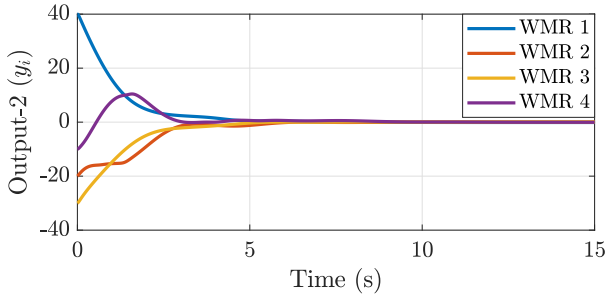


Fig. 9: Output-2 of all four WMRs

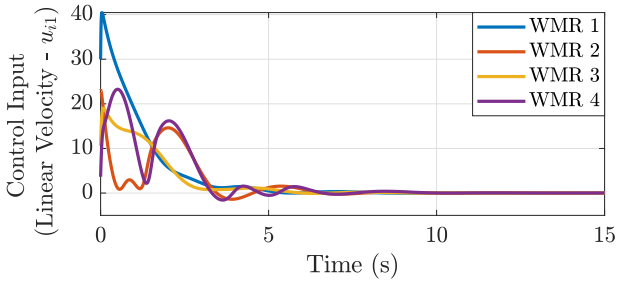


Fig. 10: Linear Velocity of all four WMRs

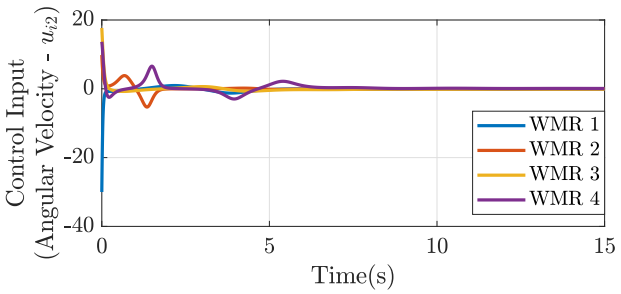


Fig. 11: Angular Velocity of all four WMRs

VI. CONCLUSIONS

The output feedback consensus problem for multiple WMRs is solved exploiting NI system properties. A strict NI property, called OSNI, is enforced on each WMR by designing distributed nonlinear state feedback control law. Retention of NI and OSNI properties are established for the proposed positive feedback control configuration through a balanced and strongly connected directed graph, which models the directional information flow among the robots. No restrictive assumption including linearization is imposed to ensure output consensus. The current work can potentially be extended toward solving the output feedback consensus problem for heterogeneous multi-robot systems possibly with non-holonomic constraints.

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