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# Containment Control of Heterogeneous Multi-agent System using Only Binary Relative Measurement

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*Abstract*— **This paper proposes a containment control law for a heterogeneous multi-agent system (MAS) by using the binary relative position measurements of the agents and the targets. Unlike the existing methods, this approach relaxes the necessity for exact position measurements as feedback to the agents, thereby the control protocol can be executed with less accurate sensors. To the best of authors' knowledge, this is the first attempt to design the containment law for a heterogeneous MAS based on the binary relative position measurements. Using non-smooth Lyapunov analysis, we establish the convergence of the proposed containment law under a detailed-balanced directed network with any positive control gain. The numerical simulations substantiate the efficacy of the proposed binary measurement-based containment law.**

## I. Introduction

Multi-agent systems (MAS) have become increasingly popular in engineering due to its inherent robustness and reliability in various cooperative missions. These advantages have led to numerous engineering applications of MAS, such as formation control [1], distributed grid management [2], multi-robot coordination [3], flocking [4], intelligent traffic management systems [5], containment control [6], and distributed optimization [7], [8]. Consensus in MAS refers to agents interacting with neighbours to reach a common reference [9]. This same concept applies to containment control problems, where agents strategically guide each other to move into the convex hull of multiple targets [10]. It has real-world applications like guiding a group of vehicles safely to their destination by avoiding obstacles [11], maintaining desired formation of multiple spacecrafts [12].

In the literature, many existing cooperative algorithms have been developed to achieve consensus [13]–[15] and containment [16], [17]. The containment control problems have been studied with homogeneous MAS where all agents possess same kind of dynamics in [18], [19] and references therein. These aforementioned works guarantee multi-agent containment under the assumptions on targets' velocity to be zero [18], constant [20], [21], or having bounded acceleration [6], [19]. During a cooperative mission, the dynamics of different deployed agents do not need to be the same in the MAS. This gives rise to the heterogeneous MAS [22], [23].

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The containment control problem for heterogeneous MAS, where each agent has linear or nonlinear dynamics, is addressed in [24]–[28] and references therein. These aforementioned works ensures accomplishment of multi-agent containment with incomplete state measurement [25] and motion uncertainty [26]. Moreover, some of them considered constraint and uncertainty in communication network as well [27], [28]. However, the above mentioned containment control laws need exact measurement of agents' position. Getting any exact measurements can become expensive in practice [29], [30]. As a result, the mission expenses increases.

To tackle this problem, researchers have developed cooperative strategies that use binary state measurements [31]. Such a strategy guarantees the desired control objectives even with limited state information, making it a cost-effective solution. In the current work, agents only know the relative position of their neighbours, which indicates $+1$ or $-1$ on each axis. They do not need the exact distance information. This simplifies the control law significantly, making them easier to implement in practice [32].

Such cooperative strategies have been studied for MAS of first [31], [33] and second order [32], [34]–[36] agents to solve consensus and cooperative tracking problems. The same concept is applied in formation control [37] and synchronization [38] problems as well. Later, the consensus protocol presented in [39] eliminates the need for velocity or exact relative state measurements of the double integrator agents, leveraging only agents' binary relative position measurements as only feedback. However, the MAS is assumed to homogeneous with double integrator agent dynamics [39]. The same protocol cannot be adapted for a group to heterogeneous agents due to the presence of discontinuity in a binary-based cooperative algorithm. Thus, it motivates to address the problem of cooperative control using agents' binary information when agents' motion model are different in orders.

This paper proposes a novel containment-consensus law for a heterogeneous MAS using the binary position measurement of the agents and the targets. The proposed protocol uses binary relative position, eliminating the need of sensors for exact position measurements. This makes the scenario simple and further reduces the burden of getting information of neighbors' state to achieve containment-consensus. Using the non-smooth Lyapunav analysis, the stability of the heterogeneous MAS of single and double integrator agents under the proposed containment law is proved with any detailed-balanced networks and positive control gains.

This work provides the following advantages and contri-

butions:

- To the best of our knowledge, the current work is the first attempt to achieve the multi-agent containment using binary relative position measurements of the agents and the targets.
- Besides, this is also the first attempt to address the containment-consensus problem for a heterogeneous MAS comprising single and double integrator agents.
- The proposed containment control law works under any detailed-balanced graph, which represents a more general network structure than an undirected network.
- Furthermore, an agent does not need the velocity information of its neighbor to execute this control law.

The rest parts of this paper are arranged in the following manner: Section II formulates the problem statement of this paper. In Section III, containment-consensus protocol for a heterogeneous MAS is presented that utilizes the binary information of relative position of the agents and targets. Section IV justifies the performance of the proposed consensus-containment protocol through numerical simulations. The findings of this work are concluded in Section V.

## II. Problem Statement and Preliminaries

Consider a heterogeneous MAS of $n$ agents consisting of $n_1$ single integrators $n_2$ double integrators, and 2 stationary targets. Their respective motion models are:

$$\left.\begin{aligned} \dot{x}_i &= u_i \quad \text{for} \quad i \in \mathbf{A}_1 \\ \dot{x}_i &= v_i \\ \dot{v}_i &= u_i \end{aligned}\right\} \text{ for } i \in \mathbf{A}_2 \tag{1}$$

where $\mathbf{A}_1 := \{A_1, A_2, \ldots, A_{n_1}\}$ is the set of single integrators and $\mathbf{A}_2 := \{A_{(n_1+1)}, A_{(n_1+2)}, \ldots, A_n\}$ is the set of double integrators. Define $\mathbf{A} := (\mathbf{A}_1 \cup \mathbf{A}_2)$ be the set of all agents and $\mathbf{T} = \{T_1, T_2\}$ is target sets. Note that $x_i \in \mathbb{R}$ denotes the position of vehicle $i \in (\mathbf{A} \cup \mathbf{T})$ and $v_i \in \mathbb{R}$ is the velocity of double integrators $i \in \mathbf{A}_2$. The control input for each agent $i \in \mathbf{A}$ is denoted as $u_i$. The minimum and maximum positions of these targets are defined as $x_m \triangleq \min_{k \in \mathbf{T}} \{x_k\}$ and $x_M \triangleq \max_{k \in \mathbf{T}} \{x_k\}$, respectively.

This work focuses on developing a distributed protocol $u_i$ to achieve the following objectives:

$$\lim_{t \to \infty} |x_i - \widehat{x}_i| = 0 \quad \text{and} \quad \lim_{t \to \infty} |\widehat{x}_i - \widehat{x}_j| = 0, \quad \forall\ i, j \in \mathbf{A}$$

where $\widehat{x}_i \in [x_m, x_M]$. This collaborative objective is referred to as the *containment-consensus* problem. Unlike a traditional containment problem, where the final positions of the agents are a weighted average of the target positions [6], the *containment-consensus* problem ensures that all agents converge to a common value within the bounds set by the positions of $T_1$ and $T_2$, that is $[x_m, x_M]$.

The proposed control law attains *containment-consensus* by using only binary relative position measurement. Such an approach eliminates the requirement for precise relative position measurements of both agents and targets. This paper considers the following assumption:

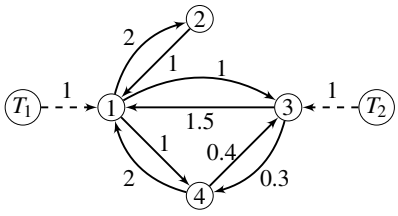



**Fig. 1:** Sensing network $\mathscr{G}_6$ with subgraph $\mathscr{G}_4$ shown in solid lines

**Assumption 1.** *The targets $T_1, T_2 \in \mathbf{T}$ are stationary.*

Next subsection presents some concepts on graph theory.

### A. Graph theory

In this paper, we assume that agents acquire binary relative position information of their neighbors through a directed sensing network, represented as a directed graph $\mathscr{G}_{n+2}$. The set of vertices $\mathscr{V}_{n+2} := (\mathbf{A} \cup \mathbf{T})$, and the set of edges $\mathscr{E}_{n+2}$ denotes the sensing connections among the agents and targets. A directed path in $\mathscr{G}$ signifies a sequence of directed edges linking consecutive nodes. For the considered sensing network, we adopt the following assumptions:

**Assumption 2.** *Only a subset of agents has the binary information of targets' relative position, while the targets have no access to the agents' state information.*

**Assumption 3.** *From each target $\{T_1, T_2\}$, there exists at least one path to each agent $i \in \mathbf{A}$.*

Assumption 2 addresses the distributed nature of the containment control in the presence of more than one target [40], [41]. Assumption 3 is necessary to ensure that the containment [40], [41].

If agent $i$ can have the binary relative position measurement of $j$, then it is labelled as the neighbor of $i$. Sensing network of an $n$ agents MAS (excluding two targets) can be represented by adjacency matrix $\mathscr{A}_n := [a_{ij}]_{n \times n}$, where $a_{ij} \begin{cases} > 0, & \text{if } j \in \mathscr{N}_i \\ = 0, & \text{otherwise} \end{cases}$ where $\mathscr{N}_i$ is the neighbor set of $i$.

A strongly connected detailed-balanced graph satisfies $w_i a_{ij} = w_j a_{ji},\ \forall\ i, j \in \mathbf{A}$ for some $\mathbf{w} \triangleq [w_1, w_2 \ldots, w_n]^\top \in \mathbb{R}^n$ such that $w_i > 0,\ \forall\ i \in \mathbf{A}$ [31], [35]. This paper assumes the following on $\mathscr{G}_n$:

**Assumption 4.** *The subgraph $\mathscr{G}_n$ representing the sensing network among n agents (excluding two targets) is a strongly connected detailed-balanced graph.*

A sensing network for a MAS with 4 agents and 2 targets is shown in Fig. 1. The dotted lines represent that Agents 1 and 3 measure binary relative positions of Targets $T_1$ and $T_2$, respectively. Solid lines depict the subgraph $\mathscr{G}_4$, the sensing network among 4 agents (excluding 2 targets). Thus, the sensing network $\mathscr{G}_6$ depicted in Fig. 1 satisfies all aforementioned Assumptions 2-4.

The proposed control law has discontinuities because of binary relative position measurement. Thus, we present a few concepts related to discontinuous systems next.

*B. Stability under discontinuity*

As discussed in [42], a system is defined with discontinuous field $f:\mathbb{R}^m \to \mathbb{R}^m$:

$$\dot{\gamma}(t) = f(\gamma(t)), \qquad \text{with } \gamma(t_0) = \gamma_0 \tag{2}$$

where $f(.)$ is measurable and essentially locally bounded function. If $f$ is essentially one-sided Lipschitz on $\mathbb{R}^m$, $\exists$ a unique solution of (2) for each $\gamma_0$ in Filippov sense, and it is same as the solution of following differential inclusion [42]:

$$\dot{\gamma} \in \mathcal{Z}(\gamma(t)) \triangleq \bigcap_{\sigma>0}\bigcap_{\mu(\mathbf{P})=0} \overline{co}\{f(B_d(\gamma,\sigma)\backslash\mathbf{P})\} \tag{3}$$

where $\mathcal{Z}(\gamma(t))$ is the set-valued map of $f(\gamma(t))$ with $\mu(\mathbf{P}) :=$ the Lebesgue measure of $\mathbf{P}$ in $\mathbb{R}^m$, a ball $B_d(\gamma,\sigma) \in \mathbb{R}^m$ centered at $\gamma$ with radius $\sigma$, and $\overline{co} :=$ the convex closure. Consider an example, $f(\gamma) = \text{sign}(\gamma) \triangleq \begin{cases} 1, & \text{for } \gamma \geq 0, \\ -1, & \text{for } \gamma < 0. \end{cases}$. Thus, according to (3), its set-valued map is $\mathcal{Z}(\gamma) = \mathbf{sgn}(\gamma) \begin{cases} = \text{sign}(\gamma), & \text{if } \gamma \neq 0, \\ \in [-1,1], & \text{if } \gamma = 0, \end{cases}$.

To establish the stability of (2), we consider a locally Lipschitz and regular function $\mathcal{V}:\mathbb{R}^m \to \mathbb{R}$ as a Lyapunov function. According to Rademacher's theorem [42], $\mathcal{V}$ is differentiable almost everywhere. Therefore, the generalized gradient of $\mathcal{V}$ at $\gamma \in \mathbb{R}^m$ is [42]:

$$\partial\mathcal{V}(\gamma) = co\left\{lim_{i\to\infty}\, d\mathcal{V}(\gamma_i) \mid \gamma_i \to \gamma,\ \gamma_i \notin \mathbf{P}\cup\Omega_f\right\} \tag{4}$$

where $\Omega_f := \{\gamma \mid \mathcal{V} \text{ is non-differentiable}\}$, $\mu(\mathbf{P}) = 0$, and $co(.) :=$ convex hull. The set-valued Lie derivative of $\mathcal{V}$ with respect to $\mathcal{Z}$ is computed as [42]:

$$\widetilde{\mathfrak{L}}_{\mathcal{Z}}\mathcal{V}(\gamma) = \left\{\vartheta \in \mathbb{R} \mid \exists\ \alpha \in \mathcal{Z} \text{ with } \beta^\top\alpha = \vartheta,\ \forall\beta \in \partial\mathcal{V}\right\}.$$

Using $\widetilde{\mathfrak{L}}_{\mathcal{Z}}\mathcal{V}(\gamma)$, the generalized version of LaSalle's invariance principle (presented next) establishes stability of (2).

**Lemma 1.** *(Theorem 4, [42]) Consider a locally Lipschitz and regular function $\mathcal{V}(\gamma):\mathbb{R}^m \to \mathbb{R}$. Define a compact set $\mathcal{H} \subset \mathbb{R}^m$ satisfying $\gamma_0 \in \mathcal{H}$ is strongly invariant for* (2)*. If $\widetilde{\mathfrak{L}}_{\mathcal{Z}}\mathcal{V}(\gamma)$ satisfies $\sup\widetilde{\mathfrak{L}}_{\mathcal{Z}}\mathcal{V}(\gamma) \leq 0$ or $\widetilde{\mathfrak{L}}_{\mathcal{Z}}\mathcal{V}(\gamma) = \emptyset\ \forall\gamma \in \mathcal{H}$, then any solution $\gamma(t):[t_0,+\infty) \to \mathbb{R}^m$ of* (2) *starting from $\gamma_0 \in \mathcal{H}$ converges to the largest weakly invariant set $\mathcal{W} \subseteq (\Omega_f \bigcap \mathcal{H})$, where $\Omega_f = \{\gamma \in \mathbb{R}^m \mid 0 \in \widetilde{\mathfrak{L}}_{\mathcal{Z}}\mathcal{V}(\gamma)\}$. In addition, if $\mathcal{W}$ is finite, the limits of all solutions with initial condition $\gamma_0$ exist and equal one of them.*

We considers the following assumption on $\Lambda:\mathbb{R}\to\mathbb{R}$:

**Assumption 5.** *A continuous odd function $\Lambda:\mathbb{R}\to\mathbb{R}$ satisfies $\Lambda(0) = 0$ and $x\Lambda(x) > 0$ for $x \neq 0$.*

Examples of such $\Lambda(x)$ are $\tanh(x)$ and $\text{sign}(x)|x|^q$, $q \in \mathbb{R}^+$. Next, containment-consensus protocol is presented for heterogenous MAS using binary relative position only.

## III. Containment-Consensus in Heterogeneous MAS

In this section, a containment control law is proposed for heterogeneous MAS of single and double integrator agents based on the binary relative position measurements of agents. The proposed containment control law for agent $i$ is

$$\left.\begin{aligned} \dot{x}_i &= -Ku_i^x && \text{for } \ i \in \mathbf{A}_1 \\ \dot{x}_i &= v_i \\ \dot{v}_i &= -\Lambda(v_i) - Ku_i^x \end{aligned}\right\} \quad \text{for } i \in \mathbf{A}_2 \tag{5}$$

where $K > 0$ and $u_i^x$ is defined as follows:

$$\begin{aligned} u_i^x = w_i \sum_{j\in\mathcal{N}_i} a_{ij}\,\text{sign}(x_i - x_j) + \frac{b_{im}}{2}\left(\text{sign}\,(x_i - x_m) - 1\right) \\ + \frac{b_{iM}}{2}\left(\text{sign}(x_i - x_M) + 1\right) \end{aligned} \tag{6}$$

Here, $a_{ij} > 0$ if $j \in \mathcal{N}_i$, $a_{ij} = 0$ for $j \notin \mathcal{N}_i$ and $w_i$ is the $i$th element of vector $\mathbf{w}$. Without loss of generality, let us assume $x_{T_1} = x_m$ and $x_{T_2} = x_M$ for this work. Thus, $x_{T_1} < x_{T_2}$ under Assumption 1. If $i$ has the binary relative position measurement of $T_1$, $b_{im} > 0$; else $b_{im} = 0$. On a similar note, when $i$ can access binary relative position measurement of target $T_2$, $b_{iM} > 0$; otherwise $b_{iM} = 0$. Refer to Fig. 1, the constants $b_{1m} = 1$, $b_{3M} = 1$, whereas $b_{im} = 0$ for $i \neq 1$ and $b_{iM} = 0$ for $i \neq 3$.

Note that $\text{sign}(x_j - x_i)$ in (6) results in the binary relative position measurement of neighbor $j$ from agent $i$. If $(x_j - x_i) \geq 0$ for some $j \in \mathcal{N}_i$, the expression $\text{sign}(x_j - x_i)$ becomes $+1$, and $\text{sign}(x_j - x_i)$ takes $-1$ value for $(x_j - x_i) < 0$. The same holds for $\text{sign}(x_m - x_i)$ and $\text{sign}(x_M - x_i)$, which represent the binary relative position measurements of targets $T_1$ and $T_2$ from the $i$th agent, respectively.

Define $\dot{x}_s := [\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_{n_1}]^\top$, $u_S^x := [{u_1}^x, {u_2}^x, \ldots, {u_{n_1}}^x]^\top \in \mathbb{R}^{n_1}$ for the set of single integrator agents, $\dot{x}_D := [\dot{x}_{(n_1+1)}, \dot{x}_{(n_1+2)}, \ldots, \dot{x}_n]^\top$, $u_D^x := [u_{(n_1+1)}^x, u_{(n_1+2)}^x, \ldots, u_n^x]^\top \in \mathbb{R}^{n_2}$, $v_D := [v_{(n_1+1)D}, v_{(n_1+2)D}, \ldots, v_{nD}]^\top$, and $\Lambda(v_D) = [\Lambda(v_{(n_1+1)D}), \Lambda(v_{(n_1+2)D}), \ldots, \Lambda(v_{nD})]^\top \in \mathbb{R}^{n_2}$ for double integrator agents. In case of the overall MAS, $x = [x_S^\top, x_D^\top]^\top$, $\dot{x} = [\dot{x}_S^\top, \dot{x}_D^\top]^\top$, and $u^x = [(u_S^x)^\top, (u_D^x)^\top]^\top \in \mathbb{R}^n$. Thus, in matrix form, (5) can be written as

$$\begin{aligned} \dot{x}_S &= -Ku_S^x \\ \dot{x}_D &= v_D \\ \dot{v}_D &= -\Lambda(v_D) - Ku_D^x \end{aligned} \tag{7}$$

The following theorem proves the accomplishment of containment-consensus in MAS under the control law (5).

**Theorem 1.** *For an heterogeneous MAS of $n_1$ single integrators, $n_2$ double integrators, and 2 targets, with respective motion model given in* (1)*, the containment law with $K > 0$ ensures $\lim_{t\to\infty}|x_i - \widehat{x}_i| = 0$, $\lim_{t\to\infty}|\widehat{x}_i - \widehat{x}_j| = 0$, $\forall i,j \in \mathbf{A}$ under Assumptions 1-5, where $\widehat{x}_i \in [x_m,\ x_M]$, $\forall i \in \mathbf{A}$.*

*Proof.* The unique Filippov solution of (7), similar to (3), can be defined as solution of the following differential inclusion:

$$\begin{aligned} \dot{x}_S &\in \mathcal{Z}(-Ku_S^x) \\ \dot{x}_D &= \mathcal{Z}(v_D) = v_D \\ \dot{v}_D &\in \mathcal{Z}(-\Lambda(v_D) - Ku_D^x) \end{aligned} \tag{8}$$

For (8), its equilibrium must satisfy the following:

$$\begin{aligned}\mathbf{0}_{n_1} &\in \mathcal{Z}(-Ku_S^x)\\ \mathbf{0}_{n_2} &= v_D\\ \mathbf{0}_{n_2} &\in \mathcal{Z}(-\Lambda(v_D)-Ku_D^x)\end{aligned} \tag{9}$$

From (9), the equilibrium must satisfy $\mathbf{0}_{n_1} \in \mathcal{Z}(u_S^x)$, $\mathbf{0}_{n_2} = v_D$, and $\mathbf{0}_{n_2} \in \mathcal{Z}(u_D^x)$. Thus, $\mathbf{0}_n \in \mathcal{Z}\left(\left[(u_S^x)^\top, (u_D^x)^\top\right]^\top\right) = \mathcal{Z}(u^x)$. Here, the vectors $\mathbf{0}_n := 0\mathbf{1}_n \in \mathbb{R}^n$. This implies $\forall i \in \mathbf{A}$,

$$\begin{aligned}0 \in \mathcal{Z}\Bigg( w_i \sum_{j=1}^{n} a_{ij}\,\mathrm{sign}(x_i-x_j) + \left[\frac{b_{im}}{2}(\mathrm{sign}(x_i-x_m)-1)\right]\\ + \left[\frac{b_{iM}}{2}(\mathrm{sign}(x_i-x_M)+1)\right]\Bigg)\end{aligned}$$

Now, multiplying both sides $[x_i - x_m]$ and taking sum over all $i \in \mathbf{A}$, we have

$$\begin{aligned}0 \in \sum_{i=1}^{n}\Bigg[(x_i-x_m)\mathcal{Z}\Bigg( w_i \sum_{j=1}^{n} a_{ij}\,\mathrm{sign}(x_i-x_j)\\ + \frac{b_{im}}{2}[\mathrm{sign}(x_i-x_m)-1] + \frac{b_{iM}}{2}[\mathrm{sign}(x_i-x_M)+1]\Bigg)\Bigg]\\ \Rightarrow 0 \in \mathcal{Z}\Bigg( \sum_{i,j=1}^{n} w_i a_{ij} x_i\,\mathrm{sign}(x_i-x_j) - x_m \sum_{i,j=1}^{n} w_i a_{ij}\,\mathrm{sign}(x_i-x_j)\\ + \sum_{i=1}^{n}\left[\frac{b_{im}}{2}\left(|x_i-x_m| - (x_i-x_m)\right)\right]\\ + \sum_{i=1}^{n}\left[\frac{b_{iM}}{2}(x_i-x_m)(\mathrm{sign}(x_i-x_M)+1)\right]\Bigg)\end{aligned}$$

From Lemma 1 of [39], $\sum_{i=1}^{n} w_i x_i \sum_{j=1}^{n} a_{ij}\,\mathrm{sign}(x_i-x_j) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i a_{ij}|x_i-x_j|$. Thus, we obtain the following:

$$\begin{aligned}0 \in \sum_{i=1}^{n}\sum_{j=1}^{n} w_i a_{ij}|x_i-x_j| + \sum_{i=1}^{n} b_{im}\left(|x_i-x_m| - (x_i-x_m)\right)\\ + \sum_{i=1}^{n} b_{iM}(x_i-x_m)(\mathbf{sgn}(x_i-x_M)+1).\end{aligned} \tag{10}$$

Here, the first two terms on the right hand side are non-negative. For the third term, the cases are as following:

- Case 1 ($x_i > x_M$): For this, $\mathbf{sgn}(x_i - x_M) = 1$ and $(x_i - x_m) > 0$. Thus, $[(x_i-x_m)(\mathbf{sgn}(x_i-x_M)+1)] > 0$.
- Case 2 ($x_i < x_m$): Since $x_i < x_m < x_M$ results in $\mathbf{sgn}(x_i - x_M) = -1$, we get $[(x_i-x_m)(\mathbf{sgn}(x_i-x_M)+1)] = 0$
- Case 3 ($x_i = x_M$): From the definition, $\mathbf{sgn}(x_i - x_M) = [-1,1] \Rightarrow (\mathbf{sgn}(x_i-x_M)+1) = [0,2]$. Thus, for $x_i = x_M > x_m$, the expression $[(x_i-x_m)(\mathbf{sgn}(x_i-x_M)+1)] \geq 0$.

Since each three terms of right hand side in (10) are non-negative, the equilibrium of (10) lies in the set $\mathcal{T} := \left\{x \in \mathbb{R}^n,\ v_D = \mathbf{0}_{n_2} \mid x_i = x_j,\ x_m \leq x_i \leq x_M,\ \forall i,j \in \mathbf{A}\right\}$. For stability analysis, consider the Lyapunov function $\mathcal{V}$: $\mathbb{R}^{n_2} \times \mathbb{R}^{n_1}$ as follows:

$$\begin{aligned}\mathcal{V} := \frac{1}{2}\sum_{i\in\mathbf{A}_2} v_i^2 + \frac{K}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} w_i a_{ij}|x_i-x_j|\\ + \frac{K}{2}\sum_{i=1}^{n} b_{im}(|x_i-x_m| - (x_i-x_m))\\ + \frac{K}{2}\sum_{i=1}^{n} b_{iM}(|x_i-x_M| + (x_i-x_M))\end{aligned} \tag{11}$$

From (11), the gradient $\partial\mathcal{V}$ calculated at $v_i$ is equal to $v_i$ $\forall i \in \mathbf{A}_2$, and $\partial\mathcal{V}$ calculated at $x_i$ is equal to $\mathcal{Z}(ku_i^x)$ $\forall i \in \mathbf{A}$. Using (8), for $\psi \in \widetilde{\mathfrak{L}}_{\mathcal{Z}}\mathcal{V}$, this satisfies the following along (5):

$$\begin{aligned}\psi \in \sum_{i\in\mathbf{A}_2} v_{iD}\mathcal{Z}\left(-\Lambda(v_{iD}) - Ku_i^x\right) + \sum_{i\in\mathbf{A}_2}\mathcal{Z}\left(Ku_i^x\right)v_{iD}\\ + \sum_{i\in\mathbf{A}_1}\mathcal{Z}\left(Ku_i^x\right)\mathcal{Z}\left(-Ku_i^x\right)\\ \Rightarrow \psi \in -\sum_{i\in\mathbf{A}_2} v_{iD}\Lambda(v_{iD}) - \sum_{i\in\mathbf{A}_1}(K)^2\mathcal{Z}(u_i^x)\mathcal{Z}(u_i^x)\end{aligned}$$

Other than the measure zero set, the trajectory must satisfy the following [43]:

$$\max(\widetilde{\mathfrak{L}}_{\mathcal{Z}}\mathcal{V}) = -\sum_{i\in\mathbf{A}_2} v_{iD}\Lambda(v_{iD}) - \sum_{i\in\mathbf{A}_1}(Ku_i^x)^2 \leq 0$$

From Lemma 1, the solution of (5) converges to the largest weakly invariant set $S = \left\{[v_D^\top,\ x_D^\top,\ x_S^\top]^\top \mid \widetilde{\mathfrak{L}}_{\mathcal{Z}}\mathcal{V} \equiv 0\right\}$. Note that, $\widetilde{\mathfrak{L}}_{\mathcal{Z}}\mathcal{V} \equiv 0 \Rightarrow v_{iD} \equiv 0,\ \forall i \in \mathbf{A}_2 \Rightarrow \dot{v}_{iD} \equiv 0,\ \forall i \in \mathbf{A}_2 \Rightarrow u_{iD}^x \equiv 0,\ \forall i \in \mathbf{A}_2$. Also, $\widetilde{\mathfrak{L}}_{\mathcal{Z}}\mathcal{V} \equiv 0 \Rightarrow u_{iS}^x \equiv 0,\ \forall i \in \mathbf{A}_1 \Rightarrow u_i^x \equiv 0,\ \forall i \in (\mathbf{A}_1 \cup \mathbf{A}_2) = \mathbf{A}$. Therefore, according to (1) the points in $S$ must satisfy the following $\forall\ i \in \mathbf{A}$:

$$\begin{aligned}0 \in \mathcal{Z}\Bigg(\Bigg[w_i \sum_{j=1}^{n} a_{ij}\,\mathrm{sign}(x_i-x_j) + b_{im}\left[\frac{1}{2}(\mathrm{sign}(x_i-x_m)-1)\right]\\ + b_{iM}\left[\frac{1}{2}(\mathrm{sign}(x_i-x_M)+1)\right]\Bigg]^\top\Bigg)\end{aligned}$$

From the earlier part of this proof, we have shown that a set $\mathcal{T} := \left\{x \in \mathbb{R}^n \mid x_i = x_j,\ x_m \leq x_i \leq x_M,\ \forall i,j \in \mathbf{A}\right\}$ satisfies the condition of the above differential inclusion. Thus, the largest invariant set in which the solution converges is $\mathcal{T}$.

Hence, we can conclude that $\lim_{t\to\infty}(x_i - x_j) = 0$ $\forall i,j \in \mathbf{A}$ where the final position $x_i \in [x_m,\ x_M], \forall i \in \mathbf{A}$. □

**Remark 1.** *The containment law can immediately be adapted for multi-dimension single and double integrator agents by using Kronecker product.*

**Remark 2.** *The earlier work on binary information-based cooperative control solves the consensus problem of a homogeneous double integrator MAS without utilizing velocity and exact relative state measurements of the double integrator agents [39]. In contrast, we have addressed the containment problem of a heterogeneous MAS using a similar strategy. Note that the existing work [39] cannot be adapted for a group of heterogeneous agents having a motion model of different orders due to the presence of discontinuity in a binary-based cooperative algorithm.*

## IV. Simulation Results

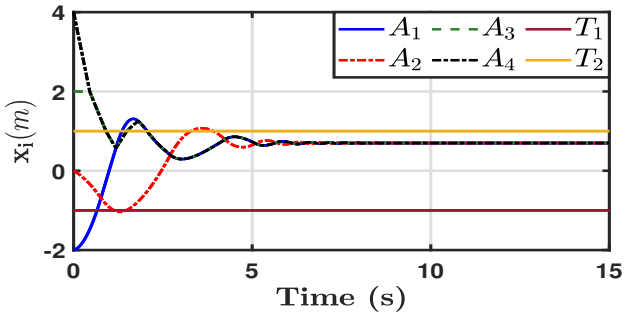
| Agents & Targets | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $T_1$ | $T_2$ |
|---|---|---|---|---|---|---|
| Position (m) | $-2$ | 0 | 2 | 4 | $-1$ | 1 |
| Velocity (m/s) | 0.25 | $-0.5$ | NA | NA | 0 | 0 |

**TABLE I:** Initial states of each agent and target

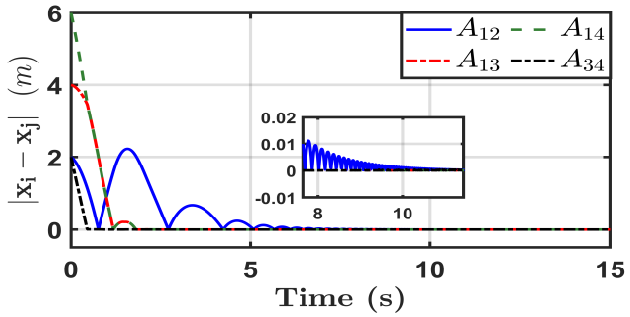
In this section, numerical simulations are presented to demonstrate the efficacy of the containment-consensus protocol developed in Section III. The proposed protocol controls a group of 4 agents to converge and stay within the convex hull created by 2 targets. The sensing graph among these agents and the targets is shown in Fig.1. For this graph, a weight vector $\mathbf{w}$ is obtained as $\left[\frac{1}{5}, \frac{1}{10}, \frac{3}{10}, \frac{2}{5}\right]^\top$. This weight vector satisfies the conditions of detailed balanced graph $w_i a_{ij} = w_j a_{ji}$, $\forall\, i, j \in \mathbf{A}$ for the considered sensor network $\mathscr{G}_6$ (as shown in Fig. 1). Table I lists the initial conditions of the agents and targets. In this example, we consider $A_1$ and $A_2$ are double integrators and $A_3$ and $A_4$ are single integrators.

For this example, we select $\Lambda(\cdot) = \tanh(\cdot)$ and set the control gain $K = 1.5$. The position trajectories of 4 agents and 2 targets are depicted in Fig. 2(a). As shown in the figure, all agents enter into the convex hull of the targets around $t = 4.5$ s. Once containment is achieved, the agents consistently remain within the boundaries formed by the targets. After containment, agents achieve consensus approximately at $t =$ 10 s. This is illustrated in Fig. 2(b) as well, where relative position of the agents are depicted. The notation $A_{ij}$ in Fig. 2(b) denotes relative measurements between agent $i$ and $j$. Figure 2(c) depicts that the velocities of double integrator agents $A_1$ and $A_2$ converges to 0 asymptotically.

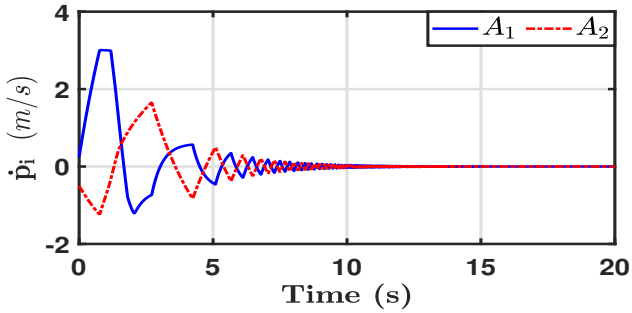
Moreover, the convergence time reduces with increasing gain $K$. Consequently, the control effort also increases. Thus, there is a trade-off while selecting the gain. The detailed discussion on the comparison with simulations is not incorporated due to the space constraint.




**(a)** Position of agents and targets




**(b)** Agents' relative position




**(c)** Agents' velocity (for $A_1$ and $A_2$)

**Fig. 2:** Agents' state trajectories to achieve containment-consensus for heterogeneous MAS using binary relative position measurement

## V. Conclusion

This current work proposes a containment-consensus law for a heterogeneous MAS that utilizes binary relative position measurements of the agents and the targets, thereby it relaxes the need for exact relative position measurements as feedback. The proposed containment control law operates effectively under a detailed-balanced network with any positive gains, which contrasts with the existing protocols relying on binary measurements that impose specific conditions on control gains to achieve control objectives aligned with the nature of the sensing graph. The stability analysis under the developed containment law is performed using the non-smooth Lyapunov analysis. Later, the numerical simulations are used to demonstrate the effectiveness of the containment control law.

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