2024 Tenth Indian Control Conference (ICC)
9 - 11 December, 2024
IISER Bhopal, India

Frequency-weighted Gramian based Truncated Balanced Realization for 2-D Separable Denominator Discrete-time Models

Deepak Kumar[1] and Kumari Kanchan[1]

***Abstract*— This paper presents a novel frequency-weighted model reduction technique for decoupled denominator-type two-dimensional discrete-time systems using truncated balanced realization that can be applied to single- and double-sided weights. The weighted controllability and observability Gramian matrices are block-diagonalized to find the transformation matrix that converts the system to a balanced form. A numerical example of a (3, 3) order first-quadrant Gaussian two-dimensional scalar filter is examined in order to demonstrate the effectiveness of the proposed method.**

I. INTRODUCTION

Many model reduction techniques have been developed for one-dimensional (1-D) systems [1], [2], over the last few decades. On the other hand, the number of techniques for reducing two-dimensional (2-D) systems is quite limited. Creating a simplified lower-order model that shares dominant attributes of the original system is the key objective of model reduction, which is achieved by minimizing a specific error between the reduced-order model (ROM) and the original system. For 2-D digital filtering, effective model reduction of 2-D systems is crucial for accomplishing faster computation. In the case of 1-D systems, truncated balanced realization (TBR) is one of the most popular and efficient model reduction techniques as it provides a closer approximation and a priori error bound. Hence, many investigators, including [3], [4], [5], [6], presented unweighted model reduction of 2-D systems using balancing-based methods.

As widely understood, no practical sensor or actuator can function across an infinite frequency range. The error between the original and ROMs must be minimized for a given frequency band in many practical applications; for example, the mathematical models of aircraft derived from physical phenomena are only applicable to a particular frequency interval. Similarly, in the case of filters, the approximation needs to be efficient within specified frequency regions. Therefore, conventional balancing-based techniques of 2-D systems like [5], which approximate the given system through the whole frequency range, might not be appropriate. The frequency-weighted 2-D model reduction problem addresses the requirement of approximation accuracy over desired frequency regions. In this sequence, Luo et al. [7], [8] solved this problem by using structured balancing, and similarly, Wang et al. [9] presented a solution by utilizing a quasi-balancing approach. In order to compute 2-D weighted controllability and observability Gramians, the systematic balancing approach [8] necessitates solving constrained convex optimization problems and 2-D Lyapunov inequalities. On the contrary, [7], [9] requires solving iterative algorithms until the required accuracy in order to assess the Gramians. Ghafoor et al. [10] presented another frequency-weighted modification that considers the modulus of the eigenvalues of the intermediate matrices. Within similarity transformation, it is found that [10] is variant, i.e., different realizations of the given model produce dissimilar outcomes. Further 2-D system model reduction improvements are shown in [11], [12], [13], which utilize linear matrix inequalities and the H_∞ approximation.

It is observed that the existing 2-D model reduction methods, which use new input-output matrices, don't affect the entries of intermediate matrices evenly, resulting in large approximation errors. Further, the other existing methods, which are based on Lyapunov inequalities and/or H_∞ approximation, make the algorithm computationally complex. Therefore, this article presents a block-diagonalization-based truncated balanced realization strategy for frequency-weighted model reduction (FWMR) to 2-D decoupled-denominator type discrete-time systems to overcome the mentioned issues. The weighting functions with input, output, or both sides may be used in the suggested method. In comparison to the current methods, the computation of weighted controllability and observability Gramians is relatively straightforward and is comparable to the one-dimensional case [8], [7], [9].

II. PRELIMINARIES

Let $G(z_1, z_2) \in \mathbb{R}^{p\times q}$ be a minimal, stable and decoupled denominator type system of order (m, n); $V(z_1, z_2) \in \mathbb{R}^{q\times t}$ and $W(z_1, z_2) \in \mathbb{R}^{s\times p}$ are minimal and stable input and output weighting functions of orders (m_v, n_v) and (m_w, n_w), respectively. We can represent $G(z_1, z_2)$, $V(z_1, z_2)$ and $W(z_1, z_2)$ using Roesser state-space form [14] as:

$$G(z_1,z_2) = C(z_1 I_m \oplus z_2 I_n - A)^{-1}B + D_*, \quad (1)$$
$$V(z_1,z_2) = C_v(z_1 I_{m_v} \oplus z_2 I_{n_v} - A_v)^{-1}B_v + D_v, \quad (2)$$
$$W(z_1,z_2) = C_w(z_1 I_{m_w} \oplus z_2 I_{n_w} - A_w)^{-1}B_w + D_w, \quad (3)$$

where

$$A = \begin{bmatrix} A_1 & A_* \\ 0 & A_4 \end{bmatrix},\ B = \begin{bmatrix} B_* \\ B_2 \end{bmatrix},\ C = \begin{bmatrix} C_1^T \\ C_*^T \end{bmatrix}^T,$$

*This work is supported by SERB-DST India, under MATRICS fellowship Grant no. MTR/2021/000722.

[1]Deepak Kumar and Kumari Kanchan are with the EED, MNNIT Allahabad, India. deepak_kumar@mnnit.ac.in

$$A_v = \begin{bmatrix} A_{v1} & A_{v2} \\ A_{v3} & A_{v4} \end{bmatrix},\ B_v = \begin{bmatrix} B_{v1} \\ B_{v2} \end{bmatrix},\ C_v = \begin{bmatrix} C_{v1}^T \\ C_{v2}^T \end{bmatrix}^T,$$

$$A_w = \begin{bmatrix} A_{w1} & A_{w2} \\ A_{w3} & A_{w4} \end{bmatrix},\ B_w = \begin{bmatrix} B_{w1} \\ B_{w2} \end{bmatrix},\ C_w = \begin{bmatrix} C_{w1}^T \\ C_{w2}^T \end{bmatrix}^T,$$

the symbol $\oplus$ denotes direct sum, $I_m, I_n, I_{m_v}, I_{n_v}, I_{m_w}$, and I_{n_w} are the identity matrices of order m, n, m_v, n_v, m_w and n_w respectively, $A \in \mathbb{R}^{(m+n)\times(m+n)}$, $B \in \mathbb{R}^{(m+n)\times q}$, $C \in \mathbb{R}^{p\times(m+n)}$, $D_* \in \mathbb{R}^{p\times q}$, $A_v \in \mathbb{R}^{(m_v+n_v)\times(m_v+n_v)}$, $B_v \in \mathbb{R}^{(m_v+n_v)\times t}$, $C_v \in \mathbb{R}^{q\times(m_v+n_v)}$, $D_v \in \mathbb{R}^{q\times t}$, $A_w \in \mathbb{R}^{(m_w+n_w)\times(m_w+n_w)}$, $B_w \in \mathbb{R}^{(m_w+n_w)\times p}$, $C_w \in \mathbb{R}^{s\times(m_w+n_w)}$ and $D_w \in \mathbb{R}^{s\times p}$.

The Roesser state-space form matrices of (1) can also be represented as [14]:

$$A = \begin{bmatrix} A_1 & 0 \\ A_* & A_4 \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ B_* \end{bmatrix},\ C = \begin{bmatrix} C_*^T \\ C_2^T \end{bmatrix}^T \quad (4)$$

The input- and output-weighted systems $G_v(z_1, z_2)$ and $G_w(z_1, z_2)$ are defined as [8]:

$$\begin{aligned} G_v(z_1, z_2) &= G(z_1, z_2)V(z_1, z_2) \\ &= \hat{C}_v(z_1 I_{m+m_v} \oplus z_2 I_{n+n_v} - \hat{A}_v)^{-1}\hat{B}_v, \end{aligned} \quad (5)$$

$$\begin{aligned} G_w(z_1, z_2) &= W(z_1, z_2)G(z_1, z_2) \\ &= \hat{C}_w(z_1 I_{m+m_w} \oplus z_2 I_{n+n_w} - \hat{A}_w)^{-1}\hat{B}_w \end{aligned} \quad (6)$$

where

$$\begin{aligned} \hat{A}_v &= \begin{bmatrix} \hat{A}_{v1} & \hat{A}_{v2} \\ \hat{A}_{v3} & \hat{A}_{v4} \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} A_1 & B_*C_{v1} & A_* & B_*C_{v2} \\ 0 & A_{v1} & 0 & A_{v2} \\ \hline 0 & B_2C_{v1} & A_4 & B_2C_{v2} \\ 0 & A_{v3} & 0 & A_{v4} \end{array}\right], \\ \hat{B}_v &= \begin{bmatrix} \hat{B}_{v1} & \hat{B}_{v2} \end{bmatrix}^T \\ &= \left[\begin{array}{cc|cc} B_*D_v & B_{v1} & B_2D_v & B_{v2} \end{array}\right]^T, \\ \hat{C}_v &= \begin{bmatrix} \hat{C}_{v1} & \hat{C}_{v2} \end{bmatrix} = \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right], \\ \hat{A}_w &= \begin{bmatrix} \hat{A}_{w1} & \hat{A}_{w2} \\ \hat{A}_{w3} & \hat{A}_{w4} \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} A_1 & 0 & A_* & 0 \\ B_{w1}C_1 & A_{w1} & B_{w1}C_* & A_{w2} \\ \hline 0 & 0 & A_4 & 0 \\ B_{w2}C_1 & A_{w3} & B_{w2}C_* & A_{w4} \end{array}\right], \\ \hat{B}_w &= \begin{bmatrix} \hat{B}_{w1} & \hat{B}_{w2} \end{bmatrix}^T \\ &= \left[\begin{array}{cc|cc} I & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{array}\right]^T, \\ \hat{C}_w &= \begin{bmatrix} \hat{C}_{w1} & \hat{C}_{w2} \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} D_wC_1 & C_{w1} & D_wC_* & C_{w2} \end{array}\right]. \end{aligned}$$

III. Proposed FWMR Method

As the given system $G(z_1, z_2)$, and weighting functions $V(z_1, z_2)$ and $W(z_1, z_2)$ are stable, it is clear that $G_v(z_1, z_2)$ and $G_w(z_1, z_2)$ are stable. Therefore, the 2-D frequency-weighted Gramians

$$\begin{aligned} \hat{P}_v &= \begin{bmatrix} \hat{P}_{v1} & \hat{P}_{v2} \\ \hat{P}_{v3} & \hat{P}_{v4} \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} \hat{P}_{v11} & \hat{P}_{v12} & \hat{P}_{v21} & \hat{P}_{v22} \\ \hat{P}_{v12}^T & \hat{P}_{v14} & \hat{P}_{v23} & \hat{P}_{v24} \\ \hline \hat{P}_{v21}^T & \hat{P}_{v23}^T & \hat{P}_{v41} & \hat{P}_{v42} \\ \hat{P}_{v22}^T & \hat{P}_{v24}^T & \hat{P}_{v42}^T & \hat{P}_{v44} \end{array}\right], \end{aligned} \quad (7)$$

$$\begin{aligned} \hat{Q}_w &= \begin{bmatrix} \hat{Q}_{w1} & \hat{Q}_{w2} \\ \hat{Q}_{w3} & \hat{Q}_{w4} \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} \hat{Q}_{w11} & \hat{Q}_{w12} & \hat{Q}_{w21} & \hat{Q}_{w22} \\ \hat{Q}_{w12}^T & \hat{Q}_{w14} & \hat{Q}_{w23} & \hat{Q}_{w24} \\ \hline \hat{Q}_{w21}^T & \hat{Q}_{w23}^T & \hat{Q}_{w41} & \hat{Q}_{w42} \\ \hat{Q}_{w22}^T & \hat{Q}_{w24}^T & \hat{Q}_{w42}^T & \hat{Q}_{w44} \end{array}\right], \end{aligned} \quad (8)$$

satisfy the following Lyapunov equations

$$\hat{A}_v\hat{P}_v\hat{A}_v^T - \hat{P}_v + \hat{B}_v\hat{B}_v^T = 0, \quad (9)$$

$$\hat{A}_w^T\hat{Q}_w\hat{A}_w - \hat{Q}_w + \hat{C}_w^T\hat{C}_w = 0. \quad (10)$$

Theorem 1: The off-diagonal sub-matrices of weighted Gramians are zero i.e., $\hat{P}_{v2} = \hat{P}_{v3} = \hat{Q}_{w2} = \hat{Q}_{w3} = 0$ for a stable, locally controllable, observable and decoupled denominator type given system $G(z_1, z_2)$, input weight $V(z_1, z_2)$ and output weight $W(z_1, z_2)$.

Proof: Follows directly from the theorem given in [4].

Let

$$\hat{T}_{v1} = \begin{bmatrix} I_m & \alpha\hat{X}_{v1} \\ 0 & I_{m_v} \end{bmatrix}, \quad (11)$$

$$\hat{T}_{v4} = \begin{bmatrix} I_n & \alpha\hat{X}_{v4} \\ 0 & I_{n_v} \end{bmatrix}, \quad (12)$$

be the transformations which block-diagonalize the (1,1) and (4,4) blocks of weighted controllability Gramian $\hat{P}_v$, we have the block-diagonalized Gramians with the following structures:

$$\begin{aligned} \hat{\hat{P}}_{v1} &= \hat{T}_{v1}^{-1}\hat{P}_{v1}\hat{T}_{v1}^{-T} \\ &= \begin{bmatrix} I_m & \alpha\hat{X}_{v1} \\ 0 & I_{m_v} \end{bmatrix}^{-1}\begin{bmatrix} \hat{P}_{v11} & \hat{P}_{v12} \\ \hat{P}_{v12}^T & \hat{P}_{v14} \end{bmatrix}\begin{bmatrix} I_m & \alpha\hat{X}_{v1} \\ 0 & I_{m_v} \end{bmatrix}^{-T} \\ &= \begin{bmatrix} \hat{P}_{v11} - \alpha^2\hat{P}_{v12}\hat{P}_{v14}^{-1}\hat{P}_{v12}^T & 0 \\ 0 & \hat{P}_{v14} \end{bmatrix}, \end{aligned} \quad (13)$$

$$\begin{aligned} \hat{\hat{P}}_{v4} &= \hat{T}_{v4}^{-1}\hat{P}_{v4}\hat{T}_{v4}^{-T} \\ &= \begin{bmatrix} I_n & \alpha\hat{X}_{v4} \\ 0 & I_{n_v} \end{bmatrix}^{-1}\begin{bmatrix} \hat{P}_{v41} & \hat{P}_{v42} \\ \hat{P}_{v42}^T & \hat{P}_{v44} \end{bmatrix}\begin{bmatrix} I_n & \alpha\hat{X}_{v1} \\ 0 & I_{n_v} \end{bmatrix}^{-T} \\ &= \begin{bmatrix} \hat{P}_{v41} - \alpha^2\hat{P}_{v42}\hat{P}_{v44}^{-1}\hat{P}_{v42}^T & 0 \\ 0 & \hat{P}_{v44} \end{bmatrix}, \end{aligned} \quad (14)$$

where $\hat{X}_{v1} = \hat{P}_{v12}\hat{P}_{v14}^{-1}, \hat{X}_{v4} = \hat{P}_{v42}\hat{P}_{v44}^{-1}$.

Similarly, the (1,1) and (4,4) blocks of weighted observability Gramian $\hat{Q}_w$ are block-diagonalized using

$$\hat{T}_{w1} = \begin{bmatrix} I_m & 0 \\ -\beta\hat{Y}_{w1} & I_{m_w} \end{bmatrix}, \quad (15)$$

$$\hat{T}_{w4} = \begin{bmatrix} I_n & 0 \\ -\beta\hat{Y}_{w4} & I_{n_w} \end{bmatrix}, \quad (16)$$

as

$$\begin{aligned}\hat{\hat{Q}}_{w1} &= \hat{T}_{w1}^T\hat{Q}_{w1}\hat{T}_{w1} \\ &= \begin{bmatrix} I_m & 0 \\ -\beta\hat{Y}_{w1} & I_{m_w} \end{bmatrix}^T \begin{bmatrix} \hat{Q}_{w11} & \hat{Q}_{w12} \\ \hat{Q}_{w12}^T & \hat{Q}_{w14} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -\beta\hat{Y}_{w1} & I_{m_w} \end{bmatrix} \\ &= \begin{bmatrix} \hat{Q}_{w11} - \beta^2\hat{Q}_{w12}^T\hat{Q}_{w14}^{-1}\hat{Q}_{w12} & 0 \\ 0 & \hat{Q}_{w14} \end{bmatrix}, \quad (17)\end{aligned}$$

$$\begin{aligned}\hat{\hat{Q}}_{w4} &= \hat{T}_{w4}^T\hat{Q}_{w4}\hat{T}_{w4} \\ &= \begin{bmatrix} I_n & 0 \\ -\beta\hat{Y}_{w4} & I_{n_w} \end{bmatrix}^T \begin{bmatrix} \hat{Q}_{w41} & \hat{Q}_{w42} \\ \hat{Q}_{w42}^T & \hat{Q}_{w44} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -\beta\hat{Y}_{w4} & I_{n_w} \end{bmatrix} \\ &= \begin{bmatrix} \hat{Q}_{w41} - \beta^2\hat{Q}_{w42}^T\hat{Q}_{w44}^{-1}\hat{Q}_{w42} & 0 \\ 0 & \hat{Q}_{w44} \end{bmatrix}, \quad (18)\end{aligned}$$

where $\hat{Y}_{w1} = \hat{Q}_{w14}^{-1}\hat{Q}_{w12}, \hat{Y}_{w4} = \hat{Q}_{w44}^{-1}\hat{Q}_{w42}$. Here, α and β are free parameters such that $0 < \alpha, \beta < 1$.

Using the minimal rank decomposition [5] of the Roesser state-space form as

$$\begin{bmatrix} A_* & B_* \\ C_* & D_* \end{bmatrix} = \begin{bmatrix} B_{1d} \\ D_{1d} \end{bmatrix} \begin{bmatrix} C_{4d} & D_{4d} \end{bmatrix}, \quad (19)$$

we can write $G(z_1, z_2) = G_{1d}(z_1)G_{4d}(z_4)$, where

$$G_{1d}(z_1) = C_1(z_1I - A_1)^{-1}B_{1d} + D_{1d}, \quad (20a)$$

$$G_{4d}(z_4) = C_{4d}(z_4I - A_4)^{-1}B_2 + D_{4d}. \quad (20b)$$

Theorem 2: The realizations $G_{1d}(z_1) = \{A_1, B_{1d}, C_1, D_{1d}\}$ and $G_{4d}(z_4) = \{A_4, B_2, C_{4d}, D_{4d}\}$ are minimal and stable.

Proof: Follows directly from the proof of unweighted case [5].

Simultaneously diagonalizing the new weighted Gramians

$$P_{ds1} = \hat{P}_{v11} - \alpha^2\hat{P}_{v12}\hat{P}_{v14}^{-1}\hat{P}_{v12}^T, \quad (21)$$

$$Q_{ds1} = \hat{Q}_{w11} - \beta^2\hat{Q}_{w12}^T\hat{Q}_{w14}^{-1}\hat{Q}_{w12} \quad (22)$$

and

$$P_{ds4} = \hat{P}_{v41} - \alpha^2\hat{P}_{v42}\hat{P}_{v44}^{-1}\hat{P}_{v42}^T, \quad (23)$$

$$Q_{ds4} = \hat{Q}_{w41} - \beta^2\hat{Q}_{w42}^T\hat{Q}_{w44}^{-1}\hat{Q}_{w42}, \quad (24)$$

we get

$$T_{1d}^{-1}P_{ds1}T_{1d}^{-T} = T_{1d}^TQ_{ds1}T_{1d}, \quad (25)$$

$$T_{4d}^{-1}P_{ds4}T_{4d}^{-T} = T_{4d}^TQ_{ds4}T_{4d}. \quad (26)$$

The weighted Gramian matrices P_{ds1}, Q_{ds1} and P_{ds4}, Q_{ds4} are obtained from the (1,1) blocks of (13),(17) and (14),(18) respectively.

Now, the realization $G_{1d}(z_1)$ and $G_{4d}(z_4)$ are transformed to $\bar{G}_{1d}(z_1) = \{\bar{A}_1, \bar{B}_1, \bar{C}_1, \bar{D}_1\}$ and $\bar{G}_{1d}(z_4) = \{\bar{A}_4, \bar{B}_4, \bar{C}_4, \bar{D}_4\}$ respectively as follows

$$\left[\begin{array}{c|c} T_{1d}^{-1}A_1T_{1d} & T_{1d}^{-1}B_{1d} \\ \hline C_1T_{1d} & D_{1d} \end{array}\right] = \left[\begin{array}{cc|c} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_{11} \\ \bar{A}_{13} & \bar{A}_{14} & \bar{B}_{12} \\ \hline \bar{C}_{11} & \bar{C}_{12} & D_{1d} \end{array}\right], \quad (27)$$

$$\left[\begin{array}{c|c} T_{4d}^{-1}A_4T_{4d} & T_{4d}^{-1}B_2 \\ \hline C_{4d}T_{4d} & D_{4d} \end{array}\right] = \left[\begin{array}{cc|c} \bar{A}_{41} & \bar{A}_{42} & \bar{B}_{41} \\ \bar{A}_{43} & \bar{A}_{44} & \bar{B}_{42} \\ \hline \bar{C}_{41} & \bar{C}_{42} & D_{4d} \end{array}\right]. \quad (28)$$

From the above realizations, the ROM is determined as

$$\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array}\right] = \left[\begin{array}{cc|c} \bar{A}_{11} & \bar{B}_{11}\bar{C}_{41} & \bar{B}_{11}D_{4d} \\ 0 & \bar{A}_{41} & \bar{B}_{41} \\ \hline \bar{C}_{11} & D_{1d}\bar{C}_{41} & D_{1d}D_{4d} \end{array}\right]. \quad (29)$$

A. Algorithm

Given $G(z_1, z_2)$, $V(z_1, z_2)$ and $W(z_1, z_2)$, the steps to use the suggested algorithm are as follows:

(i) Form input and output augmented systems using $G(z_1, z_2)$, $V(z_1, z_2)$ and $W(z_1, z_2)$.
(ii) Compute the weighted controllability and observability Gramians $\hat{P}_v$ and $\hat{Q}_w$ as stated in (7) and (8).
(iii) Partitioning (7) and (8) to obtain $\hat{P}_{v11} \in \mathbb{R}^{m\times m}$, $\hat{P}_{v41} \in \mathbb{R}^{n\times n}, \hat{Q}_{w11} \in \mathbb{R}^{m\times m}$ and $\hat{Q}_{w41} \in \mathbb{R}^{n\times n}$.
(iv) Construct the transformation matrices $\hat{T}_{v1}, \hat{T}_{v4}, \hat{T}_{w1}$ and $\hat{T}_{w4}$ using (11), (12), (15) and (16) respectively.
(v) Obtain the transformed controllability and observability Gramians from (13), (14), (17) and (18) respectively.
(v) Simultaneously diagonalize the sub-matrices P_{ds1}, Q_{ds1} and P_{ds4}, Q_{ds4} as shown in (25) and (26).
(vi) Obtain the decoupled denominator transfer functions from (19) and (20).
(vii) Transform the realizations obtained in Step (vi) using the transformation matrices T_{1d} and T_{4d} as shown in (27) and (28).
(viii) Partition the transformed realization of previous step as shown in RHS of (27) and (28) where $\bar{A}_{11} \in \mathbb{R}^{m_r\times m_r}$ and $\bar{A}_{41} \in \mathbb{R}^{n_r\times n_r}$.
(ix) From the sub-matrices of transformed realization in (27) and (28), compute the truncated model as specified in (29).

IV. SIMULATION OUTCOMES

Example 1

Consider the following decoupled-denominator type first-quadrant Gaussian two-dimensional scalar filter in the balanced form [3], [9] as

$$A = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & A_4 \end{array}\right]$$

$$A_1 = \begin{bmatrix} 0.8638 & 0.2719 & -0.0389 \\ -0.2719 & -0.5951 & 0.3607 \\ -0.0389 & -0.3607 & -0.3561 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.4293 & 0.3819 & 0.1300 \\ 0.3381 & 0.2663 & 0.1023 \end{bmatrix},$$

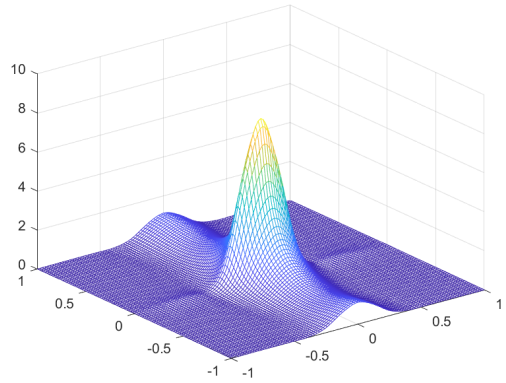


Fig. 1. Amplitude response of Example 1

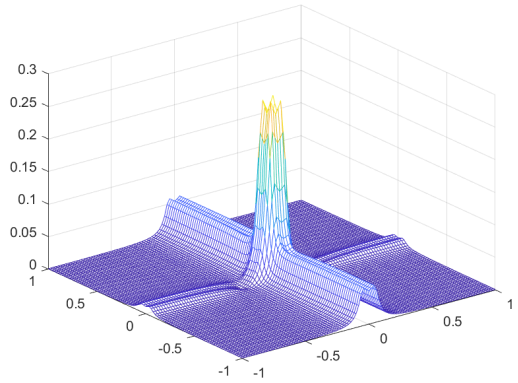


Fig. 2. Amplitude response of Input/Output weight

$$
\begin{aligned}
A_4 &= \begin{bmatrix} 0.8638 & -0.2719 & -0.0389 \\ 0.2719 & 0.5951 & -0.3607 \\ -0.0389 & 0.3607 & 0.3561 \end{bmatrix}, \\
B &= \left[\begin{array}{c|c} B_1 & B_2 \end{array}\right]^T = [\ -0.0636 \quad -0.0501 \quad -0.0192 \ | \\
&\quad -0.6547 \quad 0.5156 \quad -0.1982\ ]^T, \\
C &= \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] = [\ -0.6547 \quad 0.5156 \quad -0.1982 \ | \\
&\quad -0.06363 \quad -0.05012 \quad -0.01926\ ], \\
D &= 9.4304 \times 10^{-3}.
\end{aligned}
$$

The (2,2)-order low pass filter is used as input and output weights for the (3,3)-order system to enhance the performance over the low-frequency region, which is modelled by the following matrices

$$
\begin{aligned}
A_v &= A_w = \left[\begin{array}{c|c} A_{v1} & A_{v2} \\ \hline 0 & A_{v4} \end{array}\right] \\
&= \left[\begin{array}{cc|cc} 1.7840 & -0.8166 & 0.0177 & 0.0118 \\ 1.0000 & 0 & 0 & 0 \\ \hline 0 & 0 & 1.7840 & -0.8166 \\ 0 & 0 & 1.0000 & 0 \end{array}\right], \\
B_v &= B_w = \left[\begin{array}{c|c} B_{v1} & B_{v2} \end{array}\right]^T = \left[\begin{array}{c|c} B_{w1} & B_{w2} \end{array}\right]^T \\
&= \left[\begin{array}{cc|cc} 0.0658 & 0 & 1.0000 & 0 \end{array}\right]^T, \\
C_v &= C_w = \left[\begin{array}{c|c} C_{v1} & C_{v2} \end{array}\right] \\
&= \left[\begin{array}{cc|cc} 0.01777 & 0.01181 & 0.001169 & 7.781 \times 10^{-4} \end{array}\right], \\
D_v &= D_w = 4.3323 \times 10^{-3}.
\end{aligned}
$$

The amplitude responses of the (3,3)-order original system and the (2,2)-order input/output weight are shown in Figs. 1 and 2, which are plotted by constructing the state-space model of the original system and weight functions in terms of coefficient matrices as below

$$
G(z_1,z_2) = \begin{bmatrix} C_1 & C_* \end{bmatrix} \left[\begin{array}{c|c} z_1 I_m - A_1 & -A_* \\ \hline 0 & z_2 I_n - A_4 \end{array}\right]^{-1} \begin{bmatrix} B_* \\ B_2 \end{bmatrix},
$$

$$
V(z_1,z_2) = \begin{bmatrix} C_{v1} & C_{v2} \end{bmatrix} \left[\begin{array}{c|c} z_1 I_{m_v} - A_{v1} & -A_{v2} \\ \hline 0 & z_2 I_{n_v} - A_{v4} \end{array}\right]^{-1} \begin{bmatrix} B_{v1} \\ B_{v2} \end{bmatrix}.
$$

The (3,3)-order system is reduced to (1,1)- and (2,1)-order models by the proposed algorithm, and the amplitude responses are plotted in Figs. 4 and 6 respectively. By analyzing the responses, it is evident that the suggested approach approximates the original system quite well in the low-frequency range. Further, the amplitude responses of the modelling error transfer function $(G(z_1,z_2) - G_r(z_1,z_2))$ are shown in Figs. 3 and 5 for the (1,1)- and (2,1)-order reduced models, respectively.

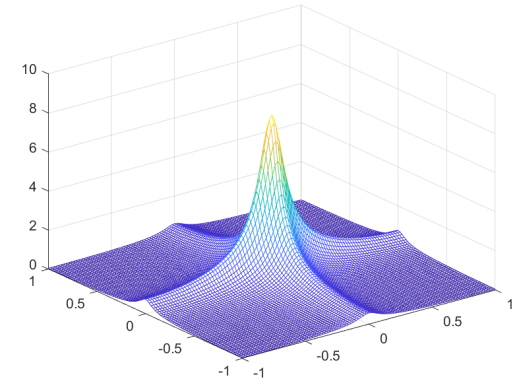


Fig. 3. Amplitude response of (1,1) order proposed ROM

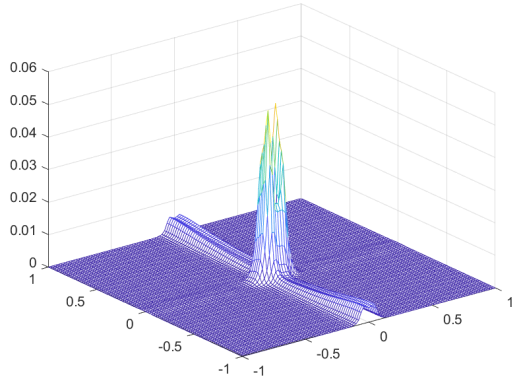


Fig. 4. Amplitude response error for (1,1) order proposed ROM

TABLE I

WEIGHTED APPROXIMATION ERROR OF ROMS

Order	Ghafoor et al. [10]	Kumar et al. [15]	Proposed		
			α	β	Error
(1, 1)	0.2968	0.0557	0.1	0.1	0.0525
(1, 2)	0.2083	0.0174	0.1	0.2	0.0265
(2, 1)	0.2892	0.0484	0.2	0.1	0.0484
(2, 2)	0.1964	0.0012	0.2	0.2	0.0020

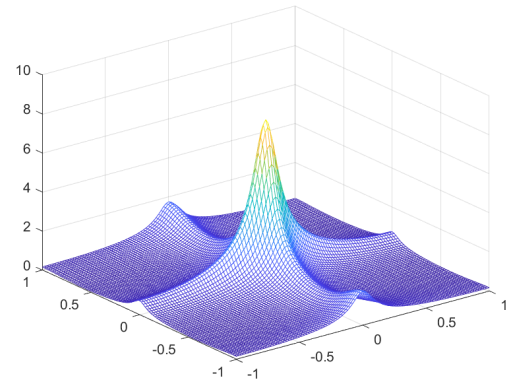


Fig. 5. Amplitude response of (2,1) order proposed ROM

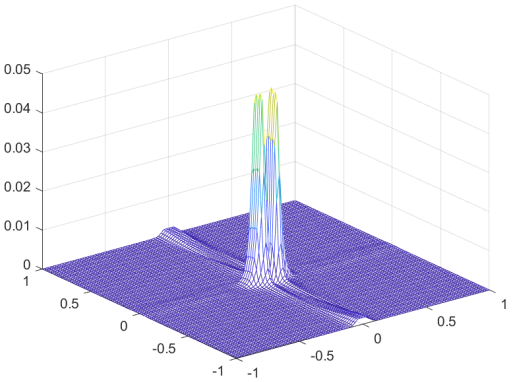


Fig. 6. Amplitude response error for (2,1) order proposed ROM

Table I shows the weighted approximation error

$$
\begin{aligned}
&||W(z_1,z_2)(G(z_1,z_2)-G_r(z_1,z_2))V(z_1,z_2)||_\infty \\
&\quad = \max_{\substack{1\le e\le E\\ 1\le f\le F}} \Big| W(e^{2\pi e/E}, e^{2\pi f/F}) \big[G(e^{2\pi e/E}, e^{2\pi f/F}) \\
&\quad - G_r(e^{2\pi e/E}, e^{2\pi f/F}) V(e^{2\pi e/E}, e^{2\pi f/F}) \big] \Big|,
\end{aligned}
$$

derived from the Ghafoor et al. [10], [15] and the proposed technique for Example 1. It is to be noted that the values of free parameters are taken randomly. We can observe from the table that the suggested method produces a good approximation of the actual system.

V. CONCLUSIONS

In this article, a new frequency-weighted truncated balanced realization is presented for 2-D decoupled denominator-type discrete-time systems to reduce the complexions involved in the computation of controllability and observability Gramians. By using the transformation matrices created with the sub-matrices of the original weighted Gramians, the suggested method block-diagonalizes the 2-D weighted Gramians to produce the new Gramians. The current work is developed focusing on 2-D systems in Roesser form as it helps to transform the unbalanced systems into a balanced form. Therefore, the presented work can be extended by considering generalized 2D systems.

REFERENCES

[1] Y. Liu and B. D. Anderson, "Singular perturbation approximation of balanced systems," *International journal of control*, vol. 50, no. 4, pp. 1379–1405, 1989.

[2] S. Gugercin and A. C. Antoulas, "A survey of model reduction by balanced truncation and some new results," *International Journal of Control*, vol. 77, no. 8, pp. 748–766, 2004.

[3] B. Lashgari, L. Silverman, and J.-F. Abramatic, "Approximation of 2-d separable in denominator filters," *IEEE transactions on circuits and systems*, vol. 30, no. 2, pp. 107–121, 1983.

[4] K. Premaratne, E. Jury, and M. Mansour, "An algorithm for model reduction of 2-d discrete time systems," *IEEE Transactions on Circuits and Systems*, vol. 37, no. 9, pp. 1116–1132, 1990.

[5] K. Zhou, Y. Li, and E. Lee, "Model reduction of 2-D systems with frequency error bounds," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 40, no. 2, pp. 107–110, 1993.

[6] K. Zhou, J. Aravena, G. Gu, and D. Xiong, "2-D model reduction by quasi-balanced truncation and singular perturbation," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 41, no. 9, pp. 593–602, 1994.

[7] H. Luo, W.-S. Lu, and A. Antoniou, "A weighted quasi-balanced model reduction method for 2-d discrete systems," in *29th Asilomar Conference on Signals, Systems and Computers*, vol. 2. IEEE, 1995, pp. 996–1000.

[8] ——, "A weighted balanced approximation for 2-d discrete systems and its application to model reduction," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 42, no. 8, pp. 419–429, 1995.

[9] D. Wang, A. Zilouchian, and Y. Bai, "An algorithm for balanced approximation and model reduction of 2-d separable-in-denominator filters," *Multidimensional Systems and Signal Processing*, vol. 16, no. 4, pp. 439–461, 2005.

[10] A. Ghafoor, W. Jing, and V. Sreeram, "Frequency-weighted model reduction method with error bounds for 2-D separable denominator discrete systems," *International Journal of Information and Systems Sciences*, vol. 1, no. 2, pp. 105–119, 2005.

[11] X. Li and H. Gao, "Min-max approximation of transfer functions with application to filter design," *IEEE Transactions on Signal Processing*, vol. 63, no. 1, pp. 31–40, 2015.

[12] X. Li, C. Yu, and H. Gao, "Frequency-limited H_∞ model reduction for positive systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 4, pp. 1093–1098, 2015.

[13] X. Li, J. Lam, and K. C. Cheung, "Generalized H_∞ model reduction for stable two-dimensional discrete systems," *Multidimensional Systems and Signal Processing*, vol. 27, no. 2, pp. 359–382, 2016.

[14] R. Roesser, "A discrete state-space model for linear image processing," *IEEE Transactions on Automatic Control*, vol. 20, no. 1, pp. 1–10, 1975.

[15] D. Kumar, U. Zulfiqar, and V. Sreeram, "Frequency-weighted balanced truncation of 2-d separable denominator discrete-time systems," in *Proc. 2018 Australian & New Zealand Control Conference (ANZCC)*. IEEE, 2018, pp. 377–379.