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Consensus in Heterogeneous Higher-Order Agents Driven by Geometric Mean under Weight-Unbalanced Digraph

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***Abstract*— This paper proposes a distributed algorithm to drive all agents in a heterogeneous multi-agent system (MAS) to the geometric mean of their initial conditions. It is defined as the *GM-consensus* problem. The proposed GM-consensus algorithm has a distributed observer for estimating the geometric mean of agents' initial states. Its later part is a local controller for tracking these observed states. This structure facilitate the scaling of the algorithm for the agents having higher-order model. The numerical example exemplifies the efficiency of the proposed distributed GM-consensus algorithm.**

***Index Terms*— Geometric mean, heterogeneous agents, higher-order integrators, multi-agent systems, weight-unbalanced graph.**

I. Introduction

The deployment of multi-agent systems (MAS) facilitate advantages to achieving common goals in different cooperative missions like formation control [1], [2], optimization [3], [4], cooperative tracking [5], sensor fusion [6]. In consensus, a significant research topic in distributed control of MAS, agents interact with their neighbors to reach a common agreement [7]. Depending upon the agreement function, the problems of consensus are identified as average consensus [8], [9], and min-/max-consensus [10], [11], and geometric mean consensus (GM-consensus) [12].

The extensive studies on different consensus problems have developed innovative and efficient algorithms, where the arithmetic mean takes the crucial role in driving the agent states into the consensus values. Moreover, it outlines the stability and convergence properties of the respective consensus algorithms [9]. The linear consensus theory is one framework for understanding linear time-varying network dynamics [7], [8]. The arithmetic mean is ideal for computing the average for datasets or system model having a linear relationship. However, such linear consensus algorithms cannot be immediately opted in the networks dynamics which are inherently non-arithmetic and need nonlinear consensus processes [12]. One notable example is the Kuramoto network model, which has applications in power grids [13] and neuroscience [14]. In such model, the chordal mean is used, which is computed for the consensus of the oscillators [15], [16]. For the existing nonlinear network model, geometric mean is one of the key tool in the problem of nonlinear dynamic averaging.

The geometric mean is most suitable for computing the average of datasets having power law relationships. Geometric mean is a versatile tool with applications in diverse domains ranging from genetics [17] and biology [18], to chemistry [19]. The geometric mean act as a feedback control gain in linearized dynamics of the gene expression networks to elucidate regulatory mechanisms and maintain stability within the network [17]. In finance section, it helps analyze portfolio performance and investment growth [20]. The logarithmic opinion pooling computes the geometric mean of many probability distributions to aggregate the averages across the network, while facilitating a balanced and coherent consensus [21]. Moreover, usage of geometric mean has potential application in distributed Bayesian filtering [22], designing algorithms [23], and detection schemes [24].

Even though geometric mean function is an important tool in understanding the stability and convergence of consensus in nonlinear network dynamics, only one significant work has been carried out in literature [12]. This work proposes a network protocol that deals with generating geometric average exponentially. The existing GM-consensus algorithms works for MAS of first-order agents under a balanced network. However, in practice, each agent can possess different motion models depending upon the physical or environmental constraints or allocated tasks [25], [26]. The existing protocol is not applicable to the heterogeneous possible higher-order MAS. Moreover, the existing work [12] requires a balanced graph to ensure GM-consensus, whereas agents' states converge to an arbitrary weighted average of the initial conditions. A weight-unbalanced graph represents the more general network structure to achieve consensus than a weight-balanced network. Due to the lack of symmetry in a weight-unbalanced graph, the design of GM-consensus protocol becomes more challenging.

All these issues motivate to develop a GM-consensus protocol for a heterogeneous MAS of higher-order integrator agents under a weight-unbalanced network. The current protocol comprises a distributed GM-observer and local tracking controller. This allows the heterogeneous MAS to achieve GM-consensus irrespective of the initial values of higher-order states.

The novelty and advantages of this work are

1) To the best of authors' knowledge the present work is the first attempt that addresses the problem of GM-consensus for a MAS of higher-order heterogeneous integrator agents.
2) The proposed protocol drives the agents' states to the GM of their initial conditions under a weight-unbalanced network. This is in contrast to [12], where the final agreement is an arbitrary weighted average of agents' initial states under same network structure.

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3) There is no assumption considered on the initial values of the higher-order derivatives.
4) Irrespective of the order of the agents in MAS, the distributed GM-observer is of order one, that facilitates its scaling for higher-order heterogeneous agents.

The rest of this paper contains the following: Section II presents requisite preliminaries on graph theory, followed by the problem statement considered in this paper. In Section III, we propose the GM-Consensus protocol for a MAS of heterogeneous, possibly higher-order integrator agents. Numerical simulations are presented in Section IV, and Section V concludes the paper.

II. Preliminaries with Problem Formulation

A. Notations

Define the following sets: $\mathbb{R}^n :=$ set of real vectors of order n, $\mathbb{R}^{n\times n} :=$ set of real matrices of order $n\times n$.

B. Graph Theory

A network of n agents is represented using a graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with set of vertices $\mathscr{V}$ representing the agents and set of edges $\mathscr{E} \subseteq (\mathscr{V}\times\mathscr{V})$ mimicking inter-agent information exchange. An edge $e_{ji} \in \mathscr{E}$ exists if i receives information from j and the corresponding neighbour set is denoted as $\mathscr{N}_i = \{j \in \mathscr{V} \mid e_{ji} \in \mathscr{E}\}$. A directed path is a sequence of consecutive edges in a digraph or directed graph. For a globally reachable node i, there exists one or more directed paths to i from the other nodes. A digraph is said to be strongly connected if and only if there exists a path between every pair of nodes in $\mathscr{G}$.

When the edges are assigned weights, it is called a weighted graph. The corresponing adjacency matrix of a graph is defined as: $\mathscr{A} := [a_{ij}]_{n\times n} \in \mathbb{R}^{n\times n}$, where if there exist an edge $e_{ji} \in \mathscr{E}$ then $a_{ij} > 0$,otherwise $a_{ij} = 0$. Using the definition of $\mathscr{A}$, the associated Laplacian matrix of the graph is defined as: $\mathscr{L} := [l_{ij}] \in \mathbb{R}^{n\times n}$ as

$$l_{ij} = \begin{cases} \sum_{j=1}^{n} a_{ij} & \text{if } i = j, \\ -a_{ij} & \text{otherwise} \end{cases}$$

For a balanced graph, the weights of $\mathscr{A}$ satisfies $\sum_{j=1}^{n} a_{ji} = \sum_{j=1}^{n} a_{ij}$ for each $i \in \mathscr{V}$ [7]. However, such assumption is relaxed for a weight-unbalanced graph. If a weight-unbalanced graph is strongly connected, then $\exists$ a $\mathbf{w} := [w_1, w_2, \ldots, w_n]^\top \in \mathbb{R}^n$ such that $w_i > 0\ \forall i$.

In this work, the underlying graph is assumed to be a weight-unbalanced strongly connected directed graph.

C. Problem Statement

In this work, we propose a consensus protocol driven by geometric mean for a MAS of 'n' heterogeneous higher-order integrator agents, each with following motion model:

$$x_i^{(d_i)} = u_i \tag{1}$$

where $u_i \in \mathbb{R}$ is the control input to agent i and $x_i \in \mathbb{R}$ denotes its position. We define $x_i^{(d_i)}$ as the d_ith time derivative of position x_i, where $d_i \in \mathbb{N}$. Since d_i can take any integer value from $\mathbb{R}$, the considered MAS consists of heterogeneous integrator agents with possibly higher-order model.

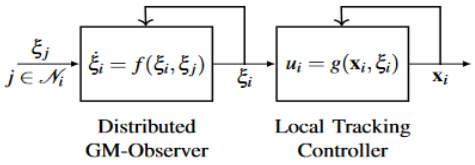


Fig. 1: Block representation of the proposed GM-consensus protocol for agent i. [$\mathscr{N}_i$:=neighbor set, ξ_i := state of GM-observer; $\mathbf{x}^\top \triangleq [x_i, \dot{x}_i, \ldots, x_i^{(d_i-1)}]^\top$, f, g :=update laws]

The objective of this work is to develop a consensus protocol for the agents with motion model (1) that ensures the following:

$$\lim_{t\to\infty} x_i(t) = \left(\prod_{i=1}^{n} x_i(t_0)\right)^{\frac{1}{n}}, \quad \lim_{t\to\infty} x_i^{(k)}(t) = 0,\ \forall k = 1, 2, \ldots, d_i.$$

where the initial positions are assumed to be positive $x_i(t_0) > 0$ for each agent. To tackle the problem of heterogeneity among the agents, we propose a distributed protocol comprising a GM-observer and a local tracking controller.

III. GM-Consensus in Heterogeneous Higher-Order Agents

This section proposed the GM-consensus protocol for the heterogeneous MAS each with motion model (1) under a strongly connected graph $\mathscr{G}$. Figure 1 represents the block diagram of the proposed GM-consensus protocol for each agent comprising a distributed GM-observer and a local tracking controller. The distributed GM-observer estimates $(\prod_{i=1}^{n} x_i(t_0))^{\frac{1}{n}}$ for agent. In the next part, the control input u_i is then designed to track the estimate of GM. Such a structure of a cooperative protocol ease the scaling of a GM-consensus protocol for different agent types, including higher-order models. Especially, in GM-consensus, certain state values of agents must remain positive through all time, which becomes challenging for higher-order agents. Our proposed cooperative framework facilitates this while ensuring GM-consensus in higher-order heterogeneous integrator agents. The forthcoming subsections detail all discussions.

A. Distributed GM-Observer

From the block diagram of the proposed GM-consensus protocol (as shown in Fig. 1), its first part is the distributed GM-observer which is given as follows:

$$\dot{\xi}_i = \log\left(\prod_{j\in\mathscr{N}_i}\left(\frac{\xi_j}{\xi_i}\right)^{a_{ij}w_i\xi_i}\right)^K \text{ with } \xi_i(t_0) = x_i(t_0) \tag{2}$$

where ξ_i is the GM-estimate of agent i, w_i is the ith element of vector $\mathbf{w}$, the constant $K > 0$ is the observer gain, and $\log(\cdot)$ denotes the natural logarithm of its argument. From (3), each agent requires the GM-estimates from its neighbors $j \in \mathscr{N}_i$ along with its own ξ_i values.

The following theorem establishes the convergence of ξ_i's to the GM of $\xi_i(t_0)$'s.

Theorem 1. *Under a strongly connected network $\mathcal{G}$, the proposed distributed GM-observer* (2) *guarantees the following:*

1) *The GM-estimate ξ_i asymptotically converges to $(\prod_{i=1}^{n} x_i(t_0))^{\frac{1}{n}}$, for each $i = 1,2,\ldots,n$.*
2) *For all time $t \geq t_0$, each GM-estimate ξ_i remains always bounded within $[\xi^m(t_0),\ \xi^M(t_0)]$, where $\xi^m(t_0) \triangleq \min_{\forall i \in \mathcal{V}}\{\xi_i(t_0)\}$ and $\xi^m(t_0) \triangleq \max_{\forall i \in \mathcal{V}}\{\xi_i(t_0)\}$.*

Proof. The proposed GM-observer (2) is rewritten as

$$\dot{\xi}_i = -w_i \sum_{j \in \mathcal{N}_i} a_{ij}\xi_i (\log \xi_i - \log \xi_j) \tag{3}$$

Consider a variable $y_i = \log \xi_i$. In terms of y_i, we can rewrite (3) in the following way:

$$\dot{y}_i = -w_i \sum_{j \in \mathcal{N}_i} a_{ij}\ (y_i - y_j) \tag{4}$$

The rest of this proof is divided two parts: first to establish the convergence of ξ_i's to $(\prod_{i=1}^{n} \xi_i(t_0))^{\frac{1}{n}}$, and then the bounds on ξ_i's are determined.

Part 1: For establishing the proof of convergence, let us define $\xi \triangleq [\xi_1, \xi_2, \ldots, \xi_n]^\top \in \mathbb{R}^n$ and $y \triangleq [y_1, y_2, \ldots, y_n]^\top \in \mathbb{R}^n$. Thus, the vector form of (4) is obtained as

$$\dot{y} = -\mathscr{W}\mathscr{L}y \tag{5}$$

where $\mathscr{L} \in \mathbb{R}^{n\times n}$ is the Laplacian matrix corresponding to the strongly connected network $\mathcal{G}$ and $\mathscr{W} := \text{diag}\{w_1, w_2, \ldots, w_n\} \in \mathbb{R}^{n\times n}$. The vector $\mathbf{w} = [w_1, w_2, \ldots, w_n]^\top$ is defined in Section II-B. In case of a strongly connected graph, $\mathscr{W}\mathscr{L}$ is equivalent to the Laplacian matrix of a weight-balanced graph [27]. Therefore, (5) ensures average consensus exponentially [7], that is for all $i = 1,2,\ldots,n$, we get

$$\lim_{t\to\infty} y_i = \frac{1}{n}\sum_{i=1}^{n} y_i(t_0).$$

Now, from the definition $y_i \triangleq \log \xi_i$, we have

$$\lim_{t\to\infty} y_i = \frac{1}{n}\sum_{i=1}^{n} y_i(t_0) \Rightarrow \lim_{t\to\infty} \log \xi_i = \frac{1}{n}\sum_{i=1}^{n} \log(\xi_i(t_0))$$

$$\Rightarrow \lim_{t\to\infty} \log \xi_i = \log\left[\left(\prod_{i=1}^{n} \xi_i(t_0)\right)^{\frac{1}{n}}\right]$$

$$\Rightarrow \lim_{t\to\infty} \xi_i = \left(\prod_{i=1}^{n} \xi_i(t_0)\right)^{\frac{1}{n}} = \left(\prod_{i=1}^{n} x_i(t_0)\right)^{\frac{1}{n}}$$

Thus, (2) guarantees GM-consensus under an strongly connected graph.

Part 2: Since the proposed GM-observer (3) has $\log(.)$ terms, it is important to assure that ξ_i corresponding to each agent must be positive $\forall t \geq t_0$. For proving the boundedness of ξ_i's, we consider the following Lyapunov function

$$V = y^M - y^m$$

where $y^M = \log \xi^M$ and $y^m = \log \xi^m$. Since natural logarithm is a monotonically increasing function, we have $y^M = \max_{\forall i \in \mathcal{V}}\{y_i\}$ and $y^m = \min_{\forall i \in \mathcal{V}}\{y_i\}$. Therefore, the Clarke's generalized derivative of V is (given in Section 2.2.8 [28])

$$\dot{V} = \dot{y}^M - \dot{y}^m = \sum_{i \in M_y} \alpha_i \dot{y}_i - \sum_{i \in m_y} \beta_i \dot{y}_i \tag{6}$$

where $\alpha_i \in (0,1]$ for $i \in M_y = \{i \mid y_i = y^M\}$ and $\beta_i \in (0,1]$ for $i \in m_y = \{i \mid y_i = y^m\}$. These constants α_i's and β_i's also satisfy the following conditions

$$\sum_{i \in M_y} \alpha_i = 1 \text{ and } \alpha_i = 0 \text{ for } i \notin M_y$$
$$\sum_{i \in m_y} \beta_i = 1 \text{ and } \beta_i = 0 \text{ for } i \notin m_y$$

For any $i \in M_y$, we have $y_j \leq y_i = y^M\ \forall\ j \in \mathcal{V}$. Thus, from (3), we obtain $\dot{y}_i$ for $i \in M_y$ as

$$\dot{y}_i = -w_i \sum_{j \in \mathcal{N}_i} a_{ij}(y_i - y_j) = -w_i \sum_{j \in \mathcal{N}_i} a_{ij}(y^M - y_j) \leq 0 \tag{7}$$

Similarly, for $i \in m_y$, we compute $\dot{y}_i$ as

$$\dot{y}_i = -w_i \sum_{j \in \mathcal{N}_i} a_{ij}(y_i - y_j) = -w_i \sum_{j \in \mathcal{N}_i} a_{ij}(y^m - y_j) \geq 0 \tag{8}$$

Therefore, from (7) and (8), the expression of $\dot{V}$ given in (6) becomes

$$\dot{V} = \sum_{i \in M_y} \alpha_i\ \dot{y}_i - \sum_{i \in m_y} \beta_i\ \dot{y}_i\ \leq\ 0$$

Thus, the Lyapunov function V is non-increasing along the trajectories (4), that is $V \leq V(t_0)\ \forall t \geq t_0$. Therefore, for each agent $i \in \mathcal{V}$, the following is satisfied:

$$y_i \in \left[y^m(t_0),\ y^M(t_0)\right],\ \forall\ t \geq t_0$$

Using the one-to-one property of natural logarithm, the following is resulted in from the definition of $y_i = \log \xi_i$:

$$\log(\xi_i) \in \left[\log(\xi^m(t_0)), \log(\xi^M(t_0))\right]\ \forall\ t \geq 0$$
$$\Rightarrow \xi_i \in \left[\xi^m(t_0)), \xi^M(t_0))\right]\ \forall\ t \geq 0$$

Hence, the boundedness on ξ_i is proved, implying $\xi_i \leq \xi_i^m(t_0) > 0\ \forall t \geq t_0$. □

The distributed GM-observer (2) guarantees asymptotic convergence of ξ_i's to their initial geometric mean. Moreover, the second part of Theorem 1 shows that ξ_i's remain positive $\forall\ t \geq t_0$. Thus, $\log \xi_i$ in (2) always give real value.

In the next section, we present a local tracking controller that solves the problem of GM-consensus for heterogeneous higher-order integrator agents.

B. Local Tracking Controller

The second part of the proposed GM-consensus protocol, a local controller to track the GM-estimate ξ_i (as shown in Fig. 1), is proposed for agent i as follows:

$$u_i = -c_1^i(x_i - \xi_i) - c_2^i\dot{x}_i - \ldots - c_{d_i}^i x_i^{(d_i-1)} \tag{9}$$

where $c^i = [c_1^i, c_1^i, \ldots, c_{(d_i-1)}^i]^\top$ is the positive controller gains which satisfies that the roots of the following equation have negative real parts [29]:

$$c_1^i + c_2^i s + c_3^i s^2 + \cdots + c_{d_i}^i s^{d_i-1} + s^{d_i} = 0$$

In this local controller (9), agent i only uses its own GM-estimate ξ_i and state measurements $\mathbf{x}_i$ (as shown in Fig. 1). Such structure of (9) eases to set the control gain c^i of each agent independently. Depending on the order an individual agent, c^i can be adjusted according to the desired performance. Moreover, the linearity in (9) makes the overall GM-consensus protocol easy to scale. The next theorem establishes the convergence of (9).

Theorem 2. *Consider a MAS of n heterogeneous integrator agents possibly of high-order, each having the motion model* (1)*. If the underlying network $\mathscr{G}$ is strongly connected, the proposed GM-consensus protocol with the distributed GM-observer* (2) *and local tracking controller* (9) *solves the problem of GM-consensus asymptotically, that is* $\lim_{t\to\infty} x_i = (\prod_{i=1}^n x_i(t_0))^{\frac{1}{n}}\ \forall i = 1,2,\ldots,n$.

Proof. Substituting the control input u_i of the agent (9) in agent's motion model (2), we get

$$\begin{aligned} x_i^{(d_i)} &= -c_1^i(x_i - \xi_i) - c_2^i\dot{x}_i - \cdots - c_{d_i}^i x_i^{(d_i-1)} \\ \Rightarrow x_i^{(d_i)} &= \left(\xi_i - \left(\prod_{i=1}^n x_i(t_0)\right)^{\frac{1}{n}}\right) - c_1^i\left(x_i - \left(\prod_{i=1}^n x_i(t_0)\right)^{\frac{1}{n}}\right) \\ &\quad - c_2^i\dot{x}_i - \cdots - c_{d_i}^i x_i^{(d_i-1)} \end{aligned}$$

From Theorem 1, $\left(\xi_i - (\prod_{i=1}^n x_i(t_0))^{\frac{1}{n}}\right)$ asymptotically converges to zero. The remaining proof is similar to the proof of Theorem 2, [10]. Therefore, the detailed proof is not incorporated in this paper. Hence, the GM-consensus protocol comprising the distributed GM-observer (2) and local tracking controller (9) solves the problem of GM-consensus asymptotically, that is as $t \to \infty$, agents' position $x_i \to (\prod_{i=1}^n x_i(t_0))^{\frac{1}{n}}$, and its time derivatives ($k \in \mathbb{N}$) satisfy $x_i^{(k)}(t) \to 0$, for each agent $i = 1,2,\ldots,n$. □

Remark 1. *For updating the distributed GM-observer* (2)*, agents exchange their estimates ξ_i's (see Fig. 1). There is no need of agents' state information* $\mathbf{x}_i = \left[x_i, \dot{x}_i, \ldots, x_i^{(d_i-1)}\right]^\top$ *in* (2)*. The same is observed in* (2) *too. Therefore, if the controller part* (9) *of certain agent is malfunctioning, it has no effect in the convergence of* (2)*. Hence, if the local controller* (9) *of an agent malfunctions, its error will not spread to other agents as long as the GM-observer* (2) *is stable. The rest of the MAS keeps working correctly. The same is verified with a numerical example in Section IV-B.*

Remark 2. *The first order structure of GM-observer* (2)*, irrespective of the order of the agents in MAS, facilitates scaling of the proposed GM-consensus protocol for higher-order heterogeneous or homogeneous integrator agents.*

Remark 3. *This work considers one dimensional agents' states, such as $x_i \in \mathbb{R}$. Since the motion of heterogeneous integrator agents can be decoupled in different dimension, the proposed protocol can immediately be applied for any multi-dimensional cases (for example, $x_i \in \mathbb{R}^d$ with $d > 1$) by using introducing Kronecker product [30].*

IV. Numerical Simulations

This section validates the proposed GM-consensus protocol through numerical examples. We consider an MAS of 5 heterogeneous integrator agent comprising 3 first-order integrators (Agents 1, 2, and 3), whereas other 2 agents (Agents 4 and 5) are second order integrators. We have considered inter-agent network to be weight-unbalanced strongly connected graph $\mathscr{G}$ as shown in Fig.2. For this graph, the left-eigenvector of the associated Laplacian matrix $\mathscr{L}$ is calculated as $\mathbf{w} = [0.298, 0.2649, 0.0795, 0.1589, 0.1987]^\top$.

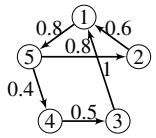


Fig. 2: Inter-agent network among 5 heterogeneous agents

Agents	1	2	3	4	5
$\xi_i(t_0)$	5	4	3	2	1
$x_i(t_0)$	5	4	3	2	1
$\dot{x}_i(t_0)$	–	–	–	−1	3

TABLE I: Agents' initial conditions

Table I describes the initial states of each agent considered in this heterogeneous MAS. In this example, the local control (9) of each agent is

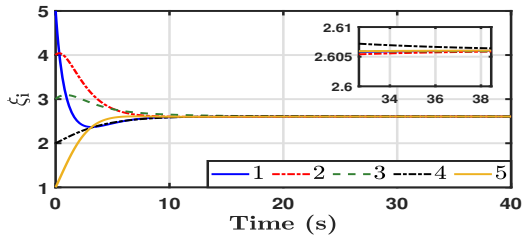
$$\begin{aligned} \dot{x}_i &= -c_1^i(x_i - \xi_i), && \text{for } i = 1,2,3 \\ \ddot{x}_i &= -c_1^i(x_i - \xi_i) - c_2^i\dot{x}_i, && \text{for } i = 4,5 \end{aligned}$$

We chose the control gain $c_1^i = 0.5$ for $i = 1,2,3$, and $c_1^i = 0.25,\ c_2^i = 1$ for $i = 4,5$.

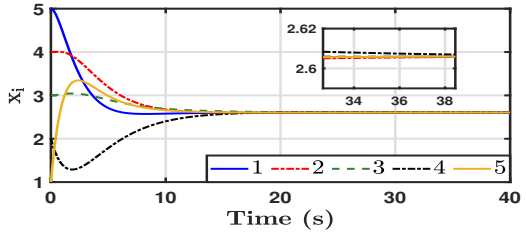
A. Performance of heterogeneous GM-consensus protocol

The states of the GM-observer ξ_i's are plotted in Fig. 3(a), which illustrates that ξ_i for each agent converges to ≈ 2.605. Thus, the GM-observer successfully estimates the geometric mean of initial positions. Moreover, the values of ξ_i's remain bounded in $[\xi^m(t_0), \xi^M(t_0)]$, which is in accordance with Theorem 1.

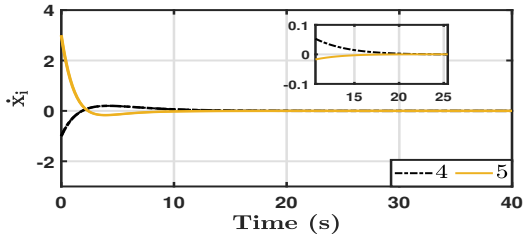
From Fig. 3(b), we observe that x_i of each agent converge to ≈ 2.605, which is the geometric mean of their initial values. The velocities $\dot{x}_i$ of the double integrators (Agents 4 and 5) also reach to 0 asymptotically. For $\pm 2\%$ error bound, the settling time of this protocol is approximately 9.5 s.



(a) GM-observer states ξ_i



(b) Agents' position



(c) Agents' velocity (for double integrators)

Fig. 3: Agents' state trajectories to achieve GM-consensus under the distributed GM-observer (2) and local controller (9)

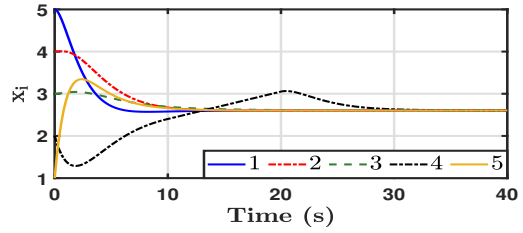


Fig. 4: Agents' positions while malfunctioning of control input

B. Robust to controller malfunction

The proposed GM-consensus protocol for the heterogeneous higher-order integrator agents is robust to malfunctioning controller as stated in Remark 1. Using same graph as in Fig.2 and initial conditions as in Table I, we have simulated the GM-consensus protocol while keeping the local controller (9) of agent 4 off for $t = 10$ to 20 s. From Fig.4, we observe that the position trajectories of agents other than 4 remain unaffected due to the malfunctioning controller of agent 4. On the other hand, the position of agent 4 deviates from its desired trajectory during the time interval $t \in [10, 20]$ s. Once the local controller (9) of Agent 4 is again turned on at $t = 20s$, it reaches to the geometric mean of agents' initial positions. Thus, such structure of proposed GM-consensus protocol eases the detection of agents with malfunctioning local controller.

V. Conclusion

In this work, we presented a cooperative protocol for a group of heterogeneous integrator agents to attain the geometric mean of their initial position. This protocol consists of a distributed observer that estimates the geometric mean of initial position and a local controller to track the individual observer state. Under a weight-unbalanced strongly connected graph, the proposed protocol solves the problem of GM-consensus asymptotically. Effectiveness of this algorithm is illustrated using numerical examples, where the position of each agent reaches to the geometric mean of their initial positions regardless of the initial values of higher state derivatives. In the future, we will investigate the problem of GM-consensus under sampled communication to make it communication efficient.

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