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Data-Driven Dissipativity Conditions for General Quadratic Supply Rate Function

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***Abstract*—The notion of dissipativity is immensely important for system analysis and controller design. In this paper, we formulate data-driven conditions for verifying dissipativity properties of a discrete linear time invariant dynamical system. Most data-driven conditions in the literature guarantee dissipativity only for a finite interval of time. Also, most data-driven methods that guarantee dissipativity consider what is referred to as the (Q, S, R) supply rate. We derive our data-driven dissipativity conditions for general supply rates which involve time differences of measured data. Moreover, our data-driven conditions guarantee dissipativity for infinite time horizon. We make use of the behavioral framework of dynamical systems and the notion of quadratic difference forms (QdFs) to formulate our data-driven dissipativity conditions as they help in generalizing the notion of dissipativity to general supply rates. Moreover, formulation of our data-driven dissipativity conditions does not require assuming persistence of excitation of measured data, which is a necessary assumption in formulation of data-driven dissipativity conditions proposed in the literature.**

I. Introduction

Due to advances in technology, modern control systems are becoming increasingly complex. Modern systems have both digital and analog components coupled together through sensors and actuators. Traditionally, for control and analysis of any dynamical system, a mathematical model of the system is required. This is referred to as the model based control paradigm [1]. But in modern systems, due to the complex coupling between various components of the system, derivation of mathematical models using first principles is no longer viable [2]. Hence, scientists nowadays are beginning to shift to data-driven methods for modelling, analysis and control dynamical systems. In data-driven methods, system model, properties, control laws, etc are inferred (or learned) using the available data. Novel data-driven approaches have been proposed various areas, like in optimal control [3], [4], predictive control [5]–[7], observer design [8], [9], etc. In recent times, an old result from [10], which was proposed for system identification, has revolutionised the field of data-driven control. This result, referred to as the *Willems' Fundamental Lemma*, allows for parameterization all possible system trajectories using a single measured or multiple measured trajectories of the system.

The fundamental lemma uses the condition of persistency of excitation of the input component of data and controllability of system to uniquely identify system model using data. In assumptions of the fundamental lemma, it is implied that one requires knowledge of which variable in the data is the input and which variable is the output. The conditions for unique system identification provided by the fundamental lemma are sufficient but not necessary. In [2], the authors formulate necessary and sufficient conditions for unique identification which do not require the assumption of persistency of excitation and controllability of the system.

The concept of dissipativity was first introduced in the seminal work by *Jan Willems* [11]. The concept of dissipativity is motivated by physical systems which satisfy energy conservation laws. In such systems, the rate of change of a so-called energy function, also referred to as storage function, is upper bounded by the energy supplied to the system. Specific nature of supply rate (conventionally defined using input-output trajectories) leads to specific types of stability results for different dynamical systems. We refer the reader to the following paper for more on this [12]. Many dissipative systems are defined by the (Q, S, R) supply rate [13] where the supply rate $s(u, y) := \begin{bmatrix} u^T & y^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$, where Q and R are symmetric matrices. But in certain cases, like in the case of negative-imaginary and counterclockwise systems [14], [15], supply rates contain time derivatives (for continuous) or time differences (for discrete) of measured input/output data.

In data-driven control, an important task is stability design for which, the knowledge about dissipativity properties of the system is important. Hence, in this paper we study the dissipativity property of a dynamical system in the data-driven context. Verifying dissipativity property in a data-driven context has been addressed before in the literature. In [16], the authors verify (Q, S, R) dissipativity, but only for finite time horizon while in [17], they verify (Q, S, R) dissipativity for infinite-time horizon. Verification of dissipativity with respect to general supply rate has been addressed in [18], where dissipativity is proved only for a finite horizon, and in [19] where authors assume a lower order of persistency of excitation of the data to guarantee dissipativity.

In all the state-of-the-art methods in the literature dealing with dissipativity verification through data-driven methods, persistency of excitation of the input component of data is assumed. Ensuring persistency of excitation implicitly assumes that an input/output partitioning of data variables is known, which is not always possible [2]. In this paper, we propose data-driven dissipativity conditions which guarantee

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dissipativity for infinite time horizon and general supply rate. We make use of the behavioral theory of dynamical systems [20] and quadratic difference forms [21] to formulate our results. More specifically:

1) We extend the results of [19] and propose data-driven dissipativity conditions (Theorem 1) for general supply rates without requiring persistency of excitation and hence knowledge of input/output partition of data variables.
2) We formulate data-driven dissipativity conditions (Corollary 1) for general supply rate with multiple shots of data.

The organization of the paper is as follows. In Section II, we discuss certain preliminaries like behavioral theory, Willems fundamental lemma, quadratic difference forms, etc. In Section III, we present the main results of this paper. And finally in Section IV, we end the paper with some final conclusions.

II. Preliminaries

In this section, we briefly review certain concepts like behavioral theory, dissipativity analysis, quadratic difference forms, fundamental lemma, etc. But first we introduce some notations used in the paper.

A. Notations

We denote the set of vectors (matrices) of order n ($n \times m$) by $\mathbb{R}^n$ ($\mathbb{R}^{n\times m}$). We denote the set of integers as $\mathbb{Z}$, real numbers as $\mathbb{R}$ and set of non-zero integers as $\mathbb{Z}_{\geqslant 0}$. Identity matrix of order n is denoted as I_n. The set of all symmetric matrices of order n is denoted as $\mathbb{S}^n$. Signal space, i.e. space of all possible trajectories of dimension $\mathbf{w}$ is denoted by $\mathbb{W}$. Any generic trajectory belonging to $\mathbb{W}$ is denoted by w. We denote the finite dimensional set of matrices with unspecified dimensions as $\mathbb{R}^{\bullet\times\bullet}$. We denote $\mathbb{R}^{m\times n}[\cdot]$ as the ring of polynomial matrices. The dual operator $*$ of a two variable polynomial matrix $\Phi(\zeta,\eta) \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta]$ is defined as: $\Phi^*(\zeta,\eta) := \Phi^T(\eta,\zeta)$. If $\Phi(\zeta,\eta) \in \mathbb{S}^{\bullet}[\zeta,\eta]$, then $\Phi = \Phi^*$

B. Behavioral Theory of Dynamical Systems

Since we consider discrete-time systems for our formulations, we restrict ourselves to discussing behavioral theory in context of discrete-time case. In the behavioral paradigm, dynamical system is characterized as a triple

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B}) \tag{1}$$

where $\mathbb{T}$ is the indexing set, $\mathbb{W}$ is the signal space with dimension $\mathbf{w}$ and $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$ is the behavior of the system. An element $w \in \mathfrak{B}$ is referred to as manifest variable. For discrete LTI systems, following requirements hold:

1) $\mathbb{T} \subset \mathcal{Z}$. In our case $\mathbb{T} = \mathbb{Z}_{\geqslant 0}$.
2) For linearity: $w_1, w_2 \in \mathfrak{B}, \implies \alpha w_1 + \beta w_2 \in \mathfrak{B}$ for all $\alpha, \beta \in \mathbb{R}$.
3) For time-invariance: $\sigma\mathfrak{B} \subset \mathfrak{B}$, where $\sigma(\cdot)$ is the shift operator, i.e. $\sigma(w(t)) = w(t+1)$ for all $t \in \mathbb{Z}_{\geqslant 0}$.

In the behavioral paradigm, a system is called controllable if two trajectories $w_1, w_2 \in \mathfrak{B}$, then it is implies that there exists $t_1, t_2 \in \mathbb{Z}_{\geqslant 0}$ such that $w_3 \in \mathfrak{B}$ where $w_3(t) = w_1(t)$ for $t \leqslant t_1$ and $w_3(t) = w_2(t - t_2)$ for $t \geqslant t_2$.

C. Dissipativity Analysis and Quadratic Difference Forms

In this section we discuss dissipativity theory from behavioral point of view. Since we are concerned with discrete-time systems, we present definitions and results for dissipativity in context of discrete-time systems only. Moreover, the definitions of dissipativity provided in this subsection is referred to as *half-line dissipativity* in the behavioral paradigm [22], [23]. We use this definition of dissipativity because in data-driven analysis, the time interval of interest is $[0, \infty)$. In the conventional definition of dissipativity, the storage function is defined using state variables while the supply rate is defined using input/output trajectories [12]. Whereas in the behavioral context, both storage function and the supply rate are expressed using manifest variables [24]. Since general supply rates contain time-differences of manifest variables, it becomes helpful to express storage functions and supply rate using quadratic difference forms (QdFs) [21]. A QdF with respect to a manifest variable w is written as:

$$Q_\Phi(w(k)) := \sum_{i=0}^{K}\sum_{j=0}^{K} w(k+i)^T \Phi_{ij} w(k+j) \tag{2}$$

If we define $\eta w(k) = w(k+1)$ and $w(k)^T\zeta = w(k+1)^T$, then equation (2) induces a two-variable polynomial $\Phi \in \mathbb{S}^{\mathbf{w}}[\zeta,\eta]$ where:

$$\Phi(\zeta,\eta) := \sum_{i=0}^{K}\sum_{j=0}^{K} \zeta^i \Phi_{ij} \eta^j \tag{3}$$

where K is called the depth of Φ. The coefficient matrix of $\Phi(\zeta,\eta)$ is:

$$\tilde{\Phi} := \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \Phi_{0K} \\ \Phi_{10} & \Phi_{11} & \cdots & \Phi_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{K0} & \Phi_{K1} & \cdots & \Phi_{KK} \end{bmatrix} \tag{4}$$

If we consider the finite length signal $\hat{w}(k) := \mathrm{col}(w(k), \dots, w(k+K))$, then the QdF $Q_\Phi(\hat{w}(k))$ can be used to express a general supply rate which contains time-differences of manifest variables. We now state the definition of dissipativity which we use for formulating data-driven conditions.

Definition 1. *[21], [22] A controllable discrete LTI system $\Sigma := (\mathbb{Z}_{\geqslant}, \mathbb{W}, \mathfrak{B})$ is dissipative with respect to supply rate defined by QdF $Q_\Phi(w)$ if there exists a storage function $Q_\Psi(w)$ such that:*

1) *For all $w \in \mathfrak{B}$ and at every $k \geqslant 0$*

$$Q_\Psi(w(k)) \geqslant 0. \tag{5}$$

2) *For all $w \in \mathfrak{B}$ and at every $k \geqslant 0$*

$$Q_\Psi(w(k+1)) - Q_\Psi(w(k)) \leqslant Q_\Phi(w(k)). \tag{6}$$

A QdF induced by Φ is called non-negative along trajectories of $\mathfrak{B}$ is represented as $\Phi \overset{\mathfrak{B}}{\geq} 0$. Thus the dissipativity condition in equations (5)-(6) can be written as [21]–[23]:

$$\Psi \overset{\mathfrak{B}}{\geq} 0 \tag{7}$$

$$\Phi - \nabla\Psi \overset{\mathfrak{B}}{\geq} 0 \tag{8}$$

where $\nabla\Psi := (\zeta\eta - 1)\Psi$.

D. Fundamental Lemma and its Extensions

In this section, we discuss Willems' Fundamental Lemma and its extension introduced in [2]. We use the indexing set $\mathbb{Z}_{\geqslant 0}$ (which includes zero) as opposed to the set of natural numbers as done in the literature just to maintain consistency with the definition of dissipativity we discussed in the last section. Let use sample a trajectory $w \in \mathfrak{B}$ for the time interval $[0, T]$. We denote the restriction of trajectory w to the interval $[0, T]$ as $w_{|T} \in (\mathbb{R}^{\mathbf{w}})^{T+1}$:

$$w_{|T} = \begin{bmatrix} w(0)^T & w(1)^T & \dots & w(T)^T \end{bmatrix}^T \tag{9}$$

We also denote the overall behavior of the system restricted to the interval $[0, T]$ as:

$$\mathfrak{B}_{|T} = \{w_{|T} | w \in \mathfrak{B}\} \tag{10}$$

Consider some $L < T$. By creating $L + 1$-length trajectories from $w_{|T}$, one can construct a Hankel matrix as:

$$\mathfrak{H}_{L+1}(w_{|T}) := \begin{bmatrix} w(0) & w(1) & \dots & w(T-L) \\ w(1) & w(2) & \dots & w(T-L+1) \\ \vdots & \vdots & \dots & \vdots \\ w(L) & w(L+1) & \dots & w(T) \end{bmatrix} \tag{11}$$

For the set of multiple trajectories $w^1, w^2, \dots w^N \in \mathfrak{B}$, restricted to interval $[0, T]$, denoted by $\mathcal{W} := \{w^1_{|T}, w^2_{|T}, \dots, w^N_{|T}\}$, we can create a mosaic Hankel matrix as:

$$\mathfrak{H}_{L+1}(\mathcal{W}) := \begin{bmatrix} \mathfrak{H}_{L+1}(w^1_{|T}) & \mathfrak{H}_{L+1}(w^2_{|T}) & \dots & \mathfrak{H}_{L+1}(w^N_{|T}) \end{bmatrix} \tag{12}$$

As mentioned in the introduction, the fundamental lemma relies on the persistence of excitation of the input component of the data. We now define what it means for a signal to be persistently exciting.

Definition 2. *Consider signal $u(t) \in (\mathbb{R}^m)^{\mathcal{T}+\infty}$, $t \in \mathbb{Z}_{\geqslant 0}$ and let u_d be the sample of $u(t)$ taken in the interval $[0, T]$. The sampled signal u_d is said to be persistently exciting of order $L + 1$, $L < T$, if*

$$rank \mathfrak{H}_{L+1}(u_d) = m(L + 1) \tag{13}$$

The formulation of the fundamental lemma is based on the concept of "Most Powerful Unfalsified Model", i.e., the model with the least complexity that conforms to the given data [25], [26]. Complexity of the system is defined using the following integer invariants. three parameters constitute the model complexity of a system:

1) **Input Cardinality** ($m(\mathfrak{B})$): This refers to maximum number of manifest variables that can be chosen at will. In the input/output representation of the given system, input cardinality refers to the number of inputs in the input/output representation.
2) **Lag** ($l(\mathfrak{B})$): This refers to the maximum degree over all possible polynomial matrix $R(s) \in \mathbb{R}^{\bullet \times q}$ such that $R(\sigma)w(t) = 0$ where $w(t) \in \mathfrak{B}$. In the input/state/output representation, lag refers to the minimum value such that the matrix $\begin{bmatrix} C^T & (CA)^T & \dots (CA^{l-1})^T \end{bmatrix}^T$ is full rank.
3) **Order of the system** ($n(\mathfrak{B})$): This refers to the degrees of freedom for the choice of the initial condition for the system. In the input/state/output representation, it refers to the number of states in the system.

We now state the fundamental lemma.

Proposition 1. *[10]For a system $\mathfrak{B} \in \mathfrak{L}^q$ with an input/output partition $w = [u^T \ y^T]^T$. Let*

1) *$w_d = [u_d^T \ y^T]^T$ be a T-length trajectory of $\mathfrak{B}$, i.e. $w_d \in \mathfrak{B}|_T$.*
2) *$\mathfrak{B}$ is controllable.*
3) *u_d is persistently exciting of order $L + n(\mathfrak{B})$, $n(\mathfrak{B})$ is the order of the system.*

Then, any L-length trajectory of $\mathfrak{B}$ can be formulated as a linear combination of the columns of $\mathfrak{H}_L(w_d)$, i.e.

$$image \ \mathfrak{H}_L(w_d) = \mathfrak{B}|_L \tag{14}$$

As concluded in [2], the fundamental lemma has certain limitations due some conditions used during its formulation. These conditions are:

1) System under consideration should be controllable.
2) Input component of the observed trajectory must be persistently exciting of a "sufficiently" high order.

Hence, in [2], the authors propose a generalized fundamental lemma which does not require the controllability and the excitation assumption. We state the result here.

Proposition 2. *[2, Theorem 17] Consider a LTI behavior $\mathfrak{B} \in \mathfrak{L}^q$ with $m(\mathfrak{B}), l(\mathfrak{B})$ and $n(\mathfrak{B})$ be the input cardinality, lag and the order of $\mathfrak{B}$ respectively. Let w_d be a $T-$length trajectory of $\mathfrak{B}$, i.e. $w_d := \begin{bmatrix} w(0)^T & \dots w(T)^T \end{bmatrix}^T$. Then $\mathfrak{B}$ is identifiable from the given trajectory w_d if and only if*

$$\text{rank } \mathfrak{H}_{L+1}(w_d) = m(\mathfrak{B})(L + 1) + n(\mathfrak{B}) \tag{15}$$

with $l < L$ and $L \leqslant T - 1$.

III. Main Results

In this section we formulate our data-driven dissipativity conditions for general supply rate. For formulating these conditions we make the following assumptions.

Assumption 1. *We assume that the parameters $m(\mathfrak{B}), l(\mathfrak{B})$ and $n(\mathfrak{B})$ are known to us.*

Assumption 2. *We assume that the system under consideration is controllable.*

Remark 1. 1) *As explained in [2], the only prior knowledge required for exact identifiability, of a system is the knowledge of input cardinality, the lag, and the order of the true system. Since we use Proposition 2 for formulating our data-riven conditions, the assumption about of knowledge of input cardinality, the lag, and the order of the true system is required. Similar assumption was also made in [17] where the lag of the system is assumed to be known. Also, in [17] the input cardinality and the number of states (i.e. the order of the system [26]) is already known from the data.*

2) *Though we assume controllability of the system, this is not explicitly required for formulating data-driven conditions. We make this assumption only because for discrete-time systems, dissipativity has been defined assuming controllability. Though there are works in the literature like [27], [28] that deal will dissipativity for uncontrollable continuous-time systems, to the best of the authors' knowledge, no such work is available in the literature for uncontrollable discrete-time systems.*

We now state our main result of the paper.

Theorem 1. *Given a controllable discrete LTI system Σ with behavior $\mathfrak{B}$ and let $w_d \in (\mathbb{R}^{\mathbf{w}})^{T+1}$ be a $T+1$-length trajectory conforming to $\mathfrak{B}$. Consider a supply rate with QdF induced by $\Phi \in \mathbb{S}^{\mathbf{w}}[\zeta,\eta]$ with depth K_Φ. Let $max(l(\mathfrak{B}), K_\Phi) < L < T$, where $l(\mathfrak{B})$ is the lag of the system. Let $rank\mathfrak{H}_{L+1}(w_d) = m(\mathfrak{B})(L+1) + n(\mathfrak{B})$, where $m(\mathfrak{B})$ be the input cardinality of $\mathfrak{B}$ and $n(\mathfrak{B})$ be its order respectively, then $\mathfrak{B}$ is dissipative with respect to the supply rate Φ if and only if there exists $\Psi \in \mathbb{S}^{\mathbf{w}}[\zeta,\eta]$ with depth K_Ψ, with $K_\Psi + 1 < L$ satisfying:*

$$\mathfrak{H}^T_{L+1}(w_d)\hat{\Psi}\mathfrak{H}_{L+1}(w_d) \geq 0 \tag{16}$$

$$\mathfrak{H}^T_{L+1}(w_d)\left(\tilde{\Phi}_{\Delta K_\Phi} - \nabla\widetilde{\Psi}_{\Delta K_\Psi}\right)\mathfrak{H}_{L+1}(w_d) \geq 0 \tag{17}$$

where

$$K_{\max} = \max\left\{K_\Phi, K_\Psi + 1\right\} \tag{18}$$

$$\Delta\Psi = L - K_\Psi \tag{19}$$

$$\Delta K_\Phi = L - K_\Phi(K_{\max} - K_\Phi) \tag{20}$$

$$\Delta K_\Psi = L - K_\Psi - 1 + (K_{\max} - K_\Psi - 1) \tag{21}$$

$$\hat{\Psi} = \mathrm{diag}(\widetilde{\Psi}, 0_{\Delta\Psi}) \tag{22}$$

$$\widetilde{\Phi}_{\Delta K_\Phi} = \mathrm{diag}\left(\widetilde{\Phi}, 0_{\Delta K_\Phi}\right) \tag{23}$$

$$\nabla\widetilde{\Psi} = \mathrm{diag}\left(0_{\mathbf{w}}, \widetilde{\Psi}\right) - \mathrm{diag}\left(\widetilde{\Psi}, 0_{\mathbf{w}}\right) \tag{24}$$

$$\nabla\widetilde{\Psi}_{\Delta K_\Psi} = \mathrm{diag}\left(\nabla\widetilde{\Psi}, 0_{\Delta K_\Psi}\right) \tag{25}$$

The proof of Theorem 1 is based on the dissipativity conditions given in equations (7)- (8) which are reformulation of the conditions in equations (5) and (6). The conditions (5) and (6) have to be satisfied for all $k \in \mathbb{Z}_{\geqslant 0}$ and for all $w \in \mathfrak{B}$. The generalized fundamental lemma (Proposition 2) states that any $L+1$-length trajectory lies in the column space of the corresponding Hankel matrix if and only if the rank condition in equation (15) is satisfied. Though, the generalized fundamental lemma does not explicitly state what should starting time instant of the $L+1$-length trajectory in the column space of the Hankel matrix satisfying (15), it is implied that due to linear and time invariance properties of a discrete LTI system, any $L+1$-trajectory starting from any time instant $k \in \mathbb{Z}_{\geqslant 0}$ will be contained in the column space of the Hankel matrix. We give a constructive proof of this fact which will we use to prove Theorem 1.

Proposition 3. *For a discrete LTI system Σ with behavior $\mathfrak{B}$ and let $w_d \in (\mathbb{R}^{\mathbf{w}})^{T+1}$ be a $T+1$-length trajectory conforming to $\mathfrak{B}$. For some $L < T$, let $\mathfrak{H}_{L+1}(w_d)$ be the Hankel matrix satisfying the rank condition in equation* (15)*. Let $\bar{w}(k) \in (\mathbb{R}^{\mathbf{w}})^{L+1}$, where $\bar{w}(k) := \begin{bmatrix} w(k)^T & \dots & w(k+L)^T \end{bmatrix}^T$ be a $L+1$-length trajectory conforming to $\mathfrak{B}$ starting at time instant $k \in \mathbb{Z}_{\geqslant 0}$. Then, $\bar{w}(k) \in$ image $\mathfrak{H}_{L+1}(w_d)$.*

Proof: Since Σ is a discrete LTI system, then for any $k \in \mathbb{Z}_{\geqslant 0}$, $\sigma^k(\mathfrak{B}) \subset \mathfrak{B}$, where $\sigma^k(\cdot)$ is the shift operator applied k-times, i.e. $\sigma^k(w(t)) = w(t+k)$. Thus for any trajectory $\begin{bmatrix} w(k)^T & w(k+1)^T & \dots \end{bmatrix}^T \in \sigma^k(\mathfrak{B})$, we have $\begin{bmatrix} w(k)^T & w(k+1)^T & \dots \end{bmatrix}^T \in \mathfrak{B}$. Let $\begin{bmatrix} w(k)^T & w(k+1)^T & \dots \end{bmatrix}^T = \begin{bmatrix} \hat{w}(0)^T & \hat{w}(1)^T & \dots \end{bmatrix}^T$. Then, by Proposition 2, we have:

$$\begin{bmatrix} w(k) \\ w(k+1) \\ \vdots \\ w(k+L) \end{bmatrix} = \begin{bmatrix} \hat{w}(0) \\ \hat{w}(1) \\ \vdots \\ \hat{w}(L) \end{bmatrix} \in \text{image } \mathfrak{H}_{L+1}(w_d). \tag{26}$$

□

We now proceed on to prove Theorem 1.

Proof: (of Theorem 1) For the proof, we represent a $L+1$-length trajectory starting at time instant $k \in \mathbb{Z}_{\geqslant 0}$ as $\hat{w}(k)_{|L+1} := \begin{bmatrix} w(k)^T & \dots & w(k+L)^T \end{bmatrix}^T$.

Only If part: Let the LTI system Σ with behavior $\mathfrak{B}$ be dissipative with respect to supply rate Φ. Then, there exists a QdF $\Psi \in \mathbb{S}^{\mathbf{w}}[\zeta,\eta]$ with $K_\Psi + 1 < L$ such that $Q_\Psi(w(k)) \geqslant 0$ for any trajectory $w \in \mathfrak{B}$ and any $k \in \mathbb{Z}_{\geqslant 0}$. Let $w(k)_{|L+1}$ be a $L+1$-length sample of w starting at time instant $k \in \mathbb{Z}_{\geqslant 0}$. Then the QdF $Q_\Psi(w(k)) = \sum_{i=0}^{K_\Psi}\sum_{j=0}^{K_\Psi} w(k+i)^T\Psi_{ij}w(k+j) = \sum_{i=0}^{L}\sum_{j=0}^{L} w(k+i)^T\Psi_{ij}w(k+j) = Q_{\bar{\Psi}}(w(k))$ where $\Psi_{ij} = 0$ for $i,j > K_\Psi$. The coefficient matrix of the Qdf $\bar{\Psi}$ will be $\hat{\Psi} = \mathrm{diag}(\widetilde{\Psi}, 0_{\Delta\Psi})$, $\Delta\Psi = L - K_\Psi$.

Then, this implies that for all $L+1$-length trajectories $\hat{w}(k)_{|L+1}$ starting at time instant $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$Q_\Psi(w(k)) = \hat{w}(k)^T_{|K_\Psi+1}\widetilde{\Psi}\hat{w}(k)_{|K_\Psi+1} = \hat{w}(k)|^T_{L+1}\hat{\Psi}\hat{w}(k)|_{L+1} \tag{27}$$

By Proposition 3, we have $\hat{w}(k)|_{L+1} \in \mathfrak{H}_{L+1}(w_d)g_k$ for some vector g_k with appropriate dimensions. Then equation (27)

and be re-written as:

$$\hat{w}(k)_{|L+1}^T \hat{\Psi} \hat{w}(k)_{|L+1} = g_k^T \mathfrak{H}_{L+1}(w_d)^T \hat{\Psi} \mathfrak{H}_{L+1}(w_d) g_k \quad (28)$$

Thus for all $w \in \mathfrak{B}$ and $k \in \mathbb{Z}_{\geqslant 0}$:

$$\hat{w}(k)_{|L+1}^T \hat{\Psi} \hat{w}(k)_{|L+1} \geqslant 0 \implies \mathfrak{H}_{L+1}(w_d)^T \hat{\Psi} \mathfrak{H}_{L+1}(w_d) \geqslant 0. \quad (29)$$

Hence, we can say that

$$Q_\Psi(w(k)) \geqslant 0 \ \forall w \in \mathfrak{B} \text{ and } k \in \mathbb{Z}_{\geqslant 0} \implies \Psi \overset{\mathfrak{B}}{\succeq} 0 \implies \mathfrak{H}_{L+1}(w_d)^T \hat{\Psi} \mathfrak{H}_{L+1}(w_d) \geqslant 0. \quad (30)$$

Now consider the QdF $Q_\Psi(w(k+1)) - Q_\Psi(w(k)))$. For any $k \in \mathbb{Z}_{\geqslant 0}$ and due to the dissipative nature of Σ, we have $Q_\Phi(w(k)) - (Q_\Psi(w(k+1) - Q_\Psi(w(k)))) \geqslant 0 \implies \Phi - \nabla\Psi \overset{\mathfrak{B}}{\succeq} 0$. The Qdf $Q_\Psi(w(k+1) - Q_\Psi(w(k))) = \sum_{i=1}^{K_\Psi+1} \sum_{j=1}^{K_\Psi+1} w(k+i)^T \Psi_{(i-1)(j-1)} w(k+j) - \sum_{i=0}^{K_\Psi} \sum_{j=0}^{K_\Psi} w(k+i)^T \Psi_{ij} w(k+j) = Q_{\nabla\Psi}(w(k))$ where the coefficient matrix of $\nabla\Psi$ is $\nabla\tilde{\Psi}$ which is defined in equation (24) and $Q_{\nabla\Psi}(w(k)) = \hat{w}(k)_{|K_\Psi+2}^T \nabla\tilde{\Psi} \hat{w}(k)_{|K_\Psi+2}$. Now the QdF $Q_\Phi(w(k)) - Q_{\nabla\Psi}(w(k))$ can be written as:

$$\begin{aligned} &Q_\Phi(w(k)) - Q_{\nabla\Psi}(w(k)) = \\ &\hat{w}(k)_{|K_\Phi+1}^T \Phi \hat{w}(k)_{|K_\Phi+1} - \hat{w}(k)_{|K_\Psi+2}^T \nabla\tilde{\Psi} \hat{w}(k)_{|K_\Psi+2} = \\ &\hat{w}(k)_{|L+1}^T \left(\tilde{\Phi}_{\Delta K_\Phi} - \nabla\widetilde{\Psi}_{\Delta K_\Psi} \right) \hat{w}(k)_{|L+1} \end{aligned} \quad (31)$$

where $\tilde{\Phi}_{\Delta K_\Phi}$ and $\nabla\widetilde{\Psi}_{\Delta K_\Psi}$ are as defined in equations (23) and (25). Hence:

$$\begin{aligned} &Q_\Phi(w(k) \geqslant Q_\Psi(w(k+1)) - Q_\Psi(w(k)) \forall k \in \mathbb{Z}_{\geqslant 0} \text{ and } w \in \mathfrak{B} \\ &\implies \Phi - \nabla\Psi \overset{\mathfrak{B}}{\succeq} 0 \implies \\ &\mathfrak{H}_{L+1}(w_d)^T \left(\bar{\Phi}_{\Delta K_\Phi} - \nabla\widetilde{\Psi}_{\Delta K_\Psi} \right) \mathfrak{H}_{L+1}(w_d) \geqslant 0. \end{aligned} \quad (32)$$

If part: Let there exists a QdF $\Psi \in \mathbb{S}^{\mathbf{w}}[\zeta, \eta]$ with depth K_ψ where $K_\Psi + 1 < L$ and consider $\mathfrak{H}_{L+1}^T(w_d) \hat{\Psi} \mathfrak{H}_{L+1}(w_d) \geqslant 0$ where $\hat{\Psi}$ defined in equations (18)- (25). By Proposition 3, for any $L+1$-length sample $\hat{w}(k)_{|L+1}$ of any trajectory $w \in \mathfrak{B}$ sampled at time instant $k \in \mathbb{Z}_{\geqslant 0}$, $\hat{w}(k)_{|L+1} = \mathfrak{H}_{L+1}(w_d) g_k$ for some vector g_k with appropriate dimensions. Thus

$$\begin{aligned} &\mathfrak{H}_{L+1}(w_d)^T \hat{\Psi} \mathfrak{H}_{L+1}(w_d) \geqslant 0 \implies \\ &\hat{w}(k)_{|L+1}^T \hat{\Psi} \hat{w}(k)_{|L+1} \geqslant 0 \text{ for all } w \in \mathfrak{B} \text{ and } k \in \mathbb{Z}_{\geqslant 0}. \end{aligned} \quad (33)$$

Thus we have:

$$\mathfrak{H}_{L+1}(w_d)^T \hat{\Psi} \mathfrak{H}_{L+1}(w_d) \implies \Psi \overset{\mathfrak{B}}{\succeq} 0 \implies Q_\Psi(w(k)) \geqslant 0 \ \forall k \in \mathbb{Z}_{\geqslant 0}, w \in \mathfrak{B}. \quad (34)$$

In a similar way, we have:

$$\mathfrak{H}_{L+1}^T(w_d) \hat{\Psi} \mathfrak{H}_{L+1}(w_d) \geqslant 0 \implies \Psi \overset{\mathfrak{B}}{\succeq} 0 \implies Q_\Phi(w(k) \geqslant Q_\Psi(w(k+1)) - Q_\Psi(w(k)) \forall k \in \mathbb{Z}_{\geqslant 0}, w \in \mathfrak{B} \quad (35)$$

and proving that Σ is dissipative with respect Φ. □

We next present a reformulation of Theorem 1 by using multiple data series. Use of multiple shots of data is advantageous because finding a single trajectory that satisfies (1) in Theorem 1 is difficult in real-life applications.

Corollary 1. *Let $\mathcal{W}_d = [w_d^1 \ w_d^2 \ \dots \ w_d^N]$ be a set of $T+1$-length trajectories, i.e., $w_d^i \in (\mathbb{R}^{\mathbf{w}})^{T+1}$, $i \in \{1, \dots, N\}$, of the LTI system Σ with behavior $\mathfrak{B}$. Consider a supply rate with QdF induced by $\Phi \in \mathbb{S}^q[\zeta, \eta]$ with depth K_Φ. Let $max(l(\mathfrak{B}), K_\Phi) < L < T$, where $l(\mathfrak{B})$ is the lag of the system. Let rank$\mathfrak{H}_{L+1}(\mathcal{W}_d) = m(\mathfrak{B})(L+1) + n(\mathfrak{B})$, where $m(\mathfrak{B})$ be the input cardinality of $\mathfrak{B}$ and $n(\mathfrak{B})$ be its order respectively, then $\mathfrak{B}$ is dissipative with respect to the supply rate Φ if and only if there exists $\Psi \in \mathbb{S}^{\mathbf{w}}[\zeta, \eta]$ with depth K_Ψ, with $K_\Psi + 1 < L$ satisfying:*

$$\mathfrak{H}_{L+1}(\mathcal{W}_d)^T \hat{\Psi} \mathfrak{H}_{L+1}(\mathcal{W}_d) \geq 0 \quad (36)$$

$$\mathfrak{H}_{L+1}(\mathcal{W}_d)^T \left(\bar{\Phi}_{\Delta K_\Phi} - \nabla\widetilde{\Psi}_{\Delta K_\Psi} \right) \mathfrak{H}_{L+1}(\mathcal{W}_d) \geq 0 \quad (37)$$

$\hat{\Psi}, \widetilde{\Phi}_{\Delta K_\Phi}$ and $\Delta\widetilde{\Psi}_{\Delta K_\Psi}$ are defined in equations (18)- (25) *and*

$$\mathfrak{H}_{L+1}(\mathcal{W}_d) = \begin{bmatrix} \mathfrak{H}_{L+1}(w_d^1) & \mathfrak{H}_{L+1}(w_d^2) & \dots & \mathfrak{H}_{L+1}(w_d^N) \end{bmatrix}$$

Proof: The condition is a reformulation of the rank condition of the generalized fundamental lemma [2, Theorem 17] for multiple shots of data [2, Corollary 21]. Thus the proof of this corollary is in the same lines as proof of Theorem 1. □

Remark 2. 1) *As mentioned in the introduction section, Theorem 1 is a generalization of [19, Theorem 3]. Our result, Theorem 1 does not require the persistency of excitation condition which was required to formulate [19, Theorem 3]. Moreover, we do not require controllability to formulate our data-driven dissipativity condition whereas controllability was required to formulate [19, Theorem 3]. We assume controllability because, as explained in Remark 1, for discrete LTI systems, dissipativity is defined assuming controllability.*

2) *Our data-driven dissipativity results (Theorem 1 and Corollary 1) verify dissipativity for infinite time horizon while results in [16] and [18] (where authors consider general supply rate) only verify dissipativity for finite time-horizon.*

3) *In [17], the authors verify dissipativity for infinite time horizon using input/state/output and input/output data and assuming persistency of excitation of the input. In our data-driven dissipativity results, since we do not require the persistency of excitation condition, this implies that we do not require the knowledge of an a-priori input/output partitioning of data variables. Moreover, in [17], the authors verify* (Q, S, R)*-dissipativity in which the the supply rate does not contain time-differences of measured data. While we formulate data-driven conditions for supply rates which contain time-differences of measured data. Our data-driven conditions are also equivalent to the ones proposed in [17, Theorem 5]. If the QdF of the supply rate* Φ *has depth equal to zero and if we know an a-priori input/output partition of data, the condition in equation* (16) *is equivalent to statements (1) and (2) in [17, Theorem 5] while equation* (17) *is equivalent of [17, Equation 16].*

IV. Conclusions

In this paper we have formulated data-driven dissipativity conditions for general supply rate. All the works in the literature make use of persistency of excitation of data for formulating data-driven dissipativity condidtions. Also, most works in the literature do not address dissipativity where the supply rate contains time-differences of measured data and verify dissipativity for finite time horizon. We formulate our data-driven dissipativity conditions by leveraging newly developed identifiability conditions developed in the behavioral framework that bypasses the need of requiring persistency of excitation of data and the knowledge of an a-priori input/output partitioning of data variable. We also makes use of the behavioral framework for discrete linear time-invariant (LTI) systems and quadratic difference forms to formulate data-driven condition for general supply rates that contain time-differences of data. We also formulate data-driven dissipativity conditions for general supply rates using multiple shots of data using the behavioral framework and QdFs. Our data-driven dissipativity conditions also guarantee dissipativity for infinite time horizon. We summarize our contributions with respect to similar works in the literature in Remark 2. For future work, we plan to use noisy data to formulate data-driven dissipativity conditions.

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