2024 Tenth Indian Control Conference (ICC)
9 - 11 December, 2024
IISER Bhopal, India

Almost Fixed-Time Consensus for Second-Order Multi-Agent System with Barrier Function Integral Sliding Mode Control

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***Abstract*—This paper addresses the leader-follower consensus (LFC) problems of multi-agent systems (MASs) having perturbed double integrator dynamics connected over a digraph. We consider perturbations in the agent dynamics to be bounded by unknown upper bounds. Integral sliding mode (ISM) control is designed to achieve robust tracking performance. We design a backstepping control to reach consensus in almost fixed time, free from singularity. A barrier function-based approach is employed to design the ISM control without information on the bound of the perturbation. We incorporate a positive definite barrier function and discuss the existence of a practical sliding mode (PSM) for the proposed scheme. We provide a simulation example to validate the proposed method.**

***Keywords*—fixed-time consensus, integral sliding mode control, multi-agent systems, directed graph, barrier-function**

I. INTRODUCTION

Recently coordinated control of multi-agent systems has gained significant attention from the research community mainly due to its potential applications in areas including coordination of unmanned aerial vehicles [1], power system [2], vehicle platoon control [3]. Convergence rate and consensus state are key concepts in this field that indicate where to go and how quickly. The consensus problems of MASs can be categorised as leaderless consensus and leader-follower consensus [4].

In the distributed coordinated control of MASs, the convergence speed is important in practical scenarios. Most of the existing studies have asymptotic or exponential notions of convergence [5]. Finite-time consensus (FNT) control [6] provides better results compared with the aforementioned notions of convergence [7]. Nonetheless, the initial conditions of the agents determine when the finite-time control rules will converge. Because the agents' initial conditions are typically unknown in advance, a predetermined convergence time cannot be guaranteed. In addition, FNT may converge to infinity due to the inappropriate initial states.

Inspired by the facts mentioned earlier, several novel fixed-time consensus (FXT) algorithms based on the concept of fixed-time stability have surfaced recently [8], where designed coordinated control laws have guaranteed consensus time independent of the initial conditions of the agents. In [9], nonlinear FXT algorithms are proposed for first and second-order integrator agents with directed communication graphs. In [10], FXT problems are addressed for the switching communication topologies.

The FXT control design for MASs has several theoretical challenges. The stability proof of FXT control is achievable for first-order dynamics [11], but obtaining theoretical results for second-order dynamics is challenging. The literature on FXT control for second-order MASs is divided into two sections. In one section, the homogeneous method is used to achieve FXT control [12], though calculating the convergence time with this method is difficult. In the other section, fixed-time nonsingular terminal SMC is introduced for FXT control in [10]. Moreover, due to the use of sign functions, most work on FXT control for disturbed MASs has resulted in discontinuous controllers. Frequent switching in real-world systems is undesirable, making a continuous control approach more advantageous for designing FXT control for disturbed second-order MASs.

The robustness and good tracking properties of sliding-mode control (SMC) have been utilized for fixed-time convergence of MASs in [13]. One major limitation of the SMC-based control design of perturbed MASs is that prior knowledge of the bounds of the disturbance is required. The barrier function (BF) is integrated into the design of adaption algorithms in SMC in [14]. Owing to the BF's dynamic properties, overestimation is prevented because the control gain drops as the disturbance gets smaller, and an upper constraint on the disturbance is no longer necessary. However, the disadvantage of the BF is that it creates an infinite control gain when the sliding variable gets closer to the domain boundary, making actuator saturation inevitable and making the implementation unfeasible. Introducing a saturation function that limits the maximum allowable gain ensures that the control input remains bounded and prevents actuator saturation [15].

In this work, we develop the fixed-time barrier function-based integral sliding mode control (FXT-BF-ISM) to solve the consensus problems of second-order MASs with bounded input disturbances. LFC control strategy is proposed where the leader has zero input. The main contributions of this manuscript are given as follows.

- A FXT control for leader-follower MASs under directed topology is proposed.
- A backstepping control is designed to achieve the consensus in almost fixed time, free from singularity for the double integrator MASs dynamics.
- A barrier function-based approach is employed to design adaptive gain of robust ISM control without information

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on the bound of the perturbations.
- The existence of a practical sliding mode is shown.

The rest of the organisation in this article is as follows. In Section II, the graph theory, technical definition, and Lemmas are described. The problem formulation is given in Section III. In Section IV the fixed-time consensus control law is proposed for the nominal system. The barrier function-based BF-ISM control for perturbed MASs is explained in Section V. Next, in Section VI, numerical simulations are given. Conclusions are stated in Section VII.

II. Preliminaries

A. Basics of graph theory

We consider the communication graph $\mathscr{G}_d = (\mathscr{V}_d, \mathscr{E}_d)$ to model the information flow among agents, where $\mathscr{E}_d \subseteq \mathscr{V}_d \times \mathscr{V}_d$ and $\mathscr{V}_d = \{1, \cdots, N\}$ denote the edge set and the vertex set, respectively. $\mathscr{N}_i$ denotes the in-neighbor set of node i, where $\mathscr{N}_i = \{j : (j,i) \in \mathscr{E}_d\}$. The adjacency matrix $\mathscr{A}_d = [a_{ij}] \in \mathbb{R}^{N\times N}$ is defined as $a_{ij} > 0$ if $(j,i) \in \mathscr{E}_d$, and $a_{ij} = 0$ otherwise, and $a_{ii} = 0$. The Laplacian matrix is defined as $\mathscr{L}_d = [l_{ij}] \in \mathbb{R}^{N\times N}$, where $l_{ii} = \sum_{j=1, j\neq i}^{k} a_{ij}$ and $l_{ij} = -a_{ij}, i \neq j$. For the leader-follower problem with one leader indexed as l, the communication between leader and followers is indicated by a pinning matrix $G = \text{diag}(g_1, \cdots, g_N) \in \mathbb{R}^{N\times N}$, with $g_i = 1$ if $l \in \mathscr{N}_i$; otherwise, $g_i = 0$. Thus $L_1 = \mathscr{L}_d + G$ is nonsingular and is diagonally dominant.

B. Technical definitions and Lemmas

We consider the following system dynamics:

$$\dot{x} = f(x); \quad x(0) = x_0, \tag{1}$$

where $x = [x_1, x_2, ..., x_n]^T \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous map satisfying $f(0) = 0$.

Definition 1: ([16] Fixed-time stability) The equilibrium of the system (1) is fixed-time stable if it is globally finite-time stable with bounded settling time T_m, where the bound is independent of any initial state, that is, for $\forall x_0$, we have $T(x_0) < T_m$, where T_m is a positive constant.

Lemma 1: [16] The equilibrium of the system (1) is fixed-time stable, if there exists a scalar dynamic system

$$\dot{V} = -\alpha V^{1-\frac{1}{m_1}} - \beta V^{1+\frac{1}{m_1}}, \tag{2}$$

where V is a positive definite function, α, β are positive constants and $m_1 > 1$. Then settling time $T(x_0)$ is bounded by

$$T(x_0) \leq T_{\text{m}} = \frac{\pi m_1}{2\sqrt{\alpha\beta}}. \tag{3}$$

Lemma 1 allows for the design of FXT control in which the predetermined consensus time depends only on the design parameters and not on the system's initial state. This suggests that choosing the right parameters will ensure the convergence time.

Lemma 2: [17] Let a vector $x \in \mathbb{R}^n$ and $q > r > 0$, then the following inequality holds

$$\|x\|_q \leq \|x\|_r \leq n^{[1/r-1/q]} \|x\|_q. \tag{4}$$

Lemma 3: [18] Since graph $\mathscr{G}_d$ is directed, the M-matrix (a matrix whose off-diagonal entries are less than or equal to zero and eigenvalues have non-negative real parts) L_1 is not necessarily symmetric. Therefore a positive diagonal matrix $E = \text{diag}(e_1, \cdots, e_N)$ exists such that $H = EL_1 + L_1^\top E$ is positive definite, where $e = [e_1, \cdots, e_N]^\top = (L_1^\top)^{-1}\mathbf{1}_N$.

III. Problem Formulation

Consider a network of N followers and one leader. The dynamics of the i^{th} follower is modelled as

$$\dot{z}_{1,i} = z_{2,i} \tag{5a}$$

$$\dot{z}_{2,i} = u_i + \Delta_i, \tag{5b}$$

where $z_{1,i} \in \mathbb{R}$ and $z_{2,i} \in \mathbb{R}, i \in \{1, \ldots, N\}$, denote the position and velocity, respectively. $u_i \in \mathbb{R}$ is the control input, and $\Delta_i(z_{1,i}, z_{2,i}, t) \in \mathbb{R}$ is the unknown uncertainty or disturbance.

The leader agent is modelled as

$$\dot{z}_{1,l} = z_{2,l} \tag{6a}$$

$$\dot{z}_{2,l} = 0, \tag{6b}$$

where $z_{1,l}, z_{2,l} \in \mathbb{R}$ are the leader's position and velocity, respectively. The control input of the leader is zero. Let us define the disagreement of the i^{th} follower as $\tilde{z}_{1,i} = z_{1,i} - z_{1,l}$ and $\tilde{z}_{2,i} = z_{2,i} - z_{2,l}$, respectively. The disagreement dynamics take the form

$$\dot{\tilde{z}}_{1,i} = \tilde{z}_{2,i} \tag{7a}$$

$$\dot{\tilde{z}}_{2,i} = u_i + \Delta_i. \tag{7b}$$

Assumption 1: The directed graph $\mathscr{G}_d$ contains a directed spanning tree with the leader as the root node.

The objective of this work is to introduce a distributed FXT control so that follower agents can track the designated leader in predefined time such that $\lim_{t\leq T_m} z_{1,i} = z_{1,l}; z_{2,i} = z_{2,l}$.

Now, we introduce the FXT-BF-ISM consensus control

$$u_i = u_{nom,i} + u_{d,i}, \tag{8}$$

where $u_{nom,i}$ is the nominal part of control and $u_{d,i}$ is the robust part of control for i^{th} follower agent. First, we will design FXT control using the backstepping technique for the nominal system, followed by an ISM-based robust control design where a barrier function will be used to determine the gain based on changes in the system's perturbations.

IV. Fixed-time Nominal Consensus Control

We treat (7) as two separate subsystems to design a controller for each one individually, and then integrate these controllers into a comprehensive control law for the entire system. A desired virtual velocity $z_{2,i}^*$ is designed for the position consensus i.e., $\tilde{z}_{1,i} = 0$. Now to ensure desired virtual velocity $z_{2,i}^*$ is equal to the real velocity $\tilde{z}_{2,i}$, i.e., error $\zeta_i = \tilde{z}_{2,i} - z_{2,i}^* = 0$, actual control u_i^* is designed.

Now, the nominal dynamics (7) with $\Delta_i = 0$ is rewritten as

$$\dot{\tilde{z}}_{1,i} = \zeta_i + z^*_{2,i} \tag{9a}$$
$$\dot{\zeta}_i = u^*_i. \tag{9b}$$

To design a distributed consensus algorithm for MASs, we define the following variables. The neighboring disagreement variables for i^{th} follower agent is given by

$$\begin{aligned}
\xi_{1,i} &= \sum_{j\in\mathcal{N}_i} a_{ij}(z_{1,i} - z_{1,j}) + g_i(z_{1,i} - z_{1,l}) \\
\xi_{2,i} &= \sum_{j\in\mathcal{N}_i} a_{ij}(z_{2,i} - z_{2,j}) + g_i(z_{2,i} - z_{2,l}) \\
\xi^*_i &= \sum_{j\in\mathcal{N}_i} a_{ij}(\zeta_i - \zeta_j) + g_i(\zeta_i - z_{2,l}),
\end{aligned} \tag{10}$$

where ξ^*_i is disagreement of error ζ_i. The above variables can be stacked as

$$\begin{aligned}
\xi_1 &= (L_1 \otimes I_n)\tilde{z}_1 \\
\xi_2 &= (L_1 \otimes I_n)\tilde{z}_2 \\
\xi^* &= (L_1 \otimes I_n)\zeta.
\end{aligned} \tag{11}$$

Now, the desired virtual velocity is designed as

$$z^*_{2,i} = \begin{cases} -\beta\xi_{1,i} & \text{if } |\xi_{1,i}| < \varepsilon \\ -\rho_3(\xi_{1,i})^{1-c/d} - \rho_4(\xi_{1,i})^{1+c/d} & \text{otherwise,} \end{cases} \tag{12}$$

where $\beta, \rho_3, \rho_4 > 0$ are positive constants and $c < d$ with $c = 2n_1$, $d = 2n_1 + 1$, where $n_1 > 0$. The parameter $\varepsilon > 0$ is a small scalar. Moreover, the actual control in FXT [17] is designed as

$$u^*_i = -\rho_1(\xi^*_i)^{1-c/d} - \rho_2(\xi^*_i)^{1+c/d} \tag{13}$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are constants. Integrating these two controllers, the fixed-time consensus protocol for nominal dynamics (7) is designed as

$$u_{nom,i} = u^*_i + \dot{z}^*_{2,i}. \tag{14}$$

Remark 1: The derivative of the intended virtual velocity in (14), involves terms like $(\xi_{1,i})^{c/d}$. When $\xi_{1,i} \to 0$, this term tends to infinity, leading to singular behavior at $\xi_{1,i} = 0$. To avoid this undesired phenomenon, we have introduced a small region $|\xi_{1,i}| < \varepsilon$ in (12), where the control is designed using a linear feedback term that ensures exponential convergence. Outside this region, the control law ensures fixed-time convergence. Thus, it can be said that the proposed scheme ensures almost fixed-time convergence.

The stability of both the subsystems (9a) and (9b) are verified in the following Theorem.

Theorem 1: Under assumption 1, the MASs dynamics (7) with the FXT protocol (14) comprised of (13) and (12), achieves consensus in almost fixed-time T_1, where $T_1 = \frac{\pi d}{c\sqrt{\alpha_1\alpha_2}} + \frac{\pi d}{c\sqrt{\alpha_3\alpha_4}}$ with

$$\alpha_1 = \rho_1 2^{-c/2d} \frac{\|H \otimes I_n\|}{(\lambda_{\max}(E))^{1-c/2d}}, \tag{15}$$

$$\alpha_2 = \rho_2 2^{c/2d}(nN)^{-c/2d} \frac{\|H \otimes I_n\|}{(\lambda_{\max}(E))^{1+c/2d}}, \tag{16}$$

$$\alpha_3 = \rho_3 2^{-c/2d} \frac{\|H \otimes I_n\|}{(\lambda_{\max}(E))^{1-c/2d}}, \tag{17}$$

$$\alpha_4 = \rho_4 2^{c/2d}(nN)^{-c/2d} \frac{\|H \otimes I_n\|}{(\lambda_{\max}(E))^{1+c/2d}}. \tag{18}$$

Proof: First, we prove the stability of the subsystem (9b), where the true velocity disagreement $\tilde{z}_{2,i}$ would match the desired virtual velocity $z^*_{2,i}$ in fixed-time independent of the initial state using virtual control u^*_i. The Lyapunov function is selected as $V_1 = \frac{1}{2}(\xi^*)^\top(E \otimes I_n)\xi^*$. Using (9) and (13), the derivative of V_1 is obtained as

$$\begin{aligned}
\dot{V}_1 =& (\xi^*)^\top(E \otimes I_n)\dot{\xi}^* = \frac{1}{2}(\xi^*)^\top(H \otimes I_n)\dot{\zeta} \\
=& \frac{1}{2}(\xi^*)^\top(H \otimes I_n)\{-\rho_1(\xi^*)^{1-c/d} - \rho_2(\xi^*)^{1+c/d}\} \\
=& -\frac{\rho_1}{2}(\xi^*)^\top(H \otimes I_n)(H \otimes I_n)^{-1+c/d}((H \otimes I_n)\xi^*)^{1-c/d} \\
& - \frac{\rho_2}{2}(\xi^*)^\top(H \otimes I_n)(H \otimes I_n)^{-1-c/d}((H \otimes I_n)\xi^*)^{1+c/d}. \\
=& -\frac{\rho_1}{2}\|(H \otimes I_n)\|^{-1+c/d}_{-1+c/d}\|(H \otimes I_n)\xi^*\|^{2-c/d}_{2-c/d} \\
& - \frac{\rho_2}{2}\|(H \otimes I_n)\|^{-1-c/d}_{-1-c/d}\|(H \otimes I_n)\xi^*\|^{2+c/d}_{2+c/d}.
\end{aligned} \tag{19}$$

Using Lemma 2, the following inequalities hold

$$\|(H \otimes I_n)\xi^*\|^{2-c/d}_{2-c/d} \geq \|(H \otimes I_n)\xi^*\|^{2-c/d}_2, \tag{20}$$

and

$$\begin{aligned}
\|(H \otimes I_n)\xi^*\|^{2+c/d}_{2+c/d} \geq& \left[(nN)^{\left(\frac{1}{2+c/d} - \frac{1}{2}\right)}\right]^{-(2+c/d)} \\
& \times \|(H \otimes I_n)\xi^*\|^{2+c/d}_2.
\end{aligned} \tag{21}$$

Using (20) and (21), we bound (19) as

$$\begin{aligned}
\dot{V}_1 \leq& -\frac{\rho_1}{2}\|(H \otimes I_n)\|^{-1+c/d}_2\|(H \otimes I_n)\xi^*\|^{2-c/d}_2 \\
& - \frac{\rho_2}{2}\|(H \otimes I_n)\|^{-1-c/d}_2(nN)^{-c/2d}\|(H \otimes I_n)\xi^*\|^{2+c/d}_2 \\
\leq& -\frac{\rho_1}{2}\|(H \otimes I_n)\|_2\|\xi^*\|^{2-c/d}_2 \\
& - \frac{\rho_2}{2}\|(H \otimes I_n)\|_2(nN)^{-c/2d}\|\xi^*\|^{2+c/d}_2.
\end{aligned} \tag{22}$$

Using $\|\xi^*\|^2 \geq \frac{2V_1}{\lambda_{\max}(E)}$, we get

$$\begin{aligned}
\dot{V}_1 \leq& -\rho_1 2^{-c/2d}\frac{\|H \otimes I_n\|}{(\lambda_{\max}(E))^{1-c/2d}}V_1^{1-c/2d} \\
& - \rho_2 2^{c/2d}(nN)^{-c/2d}\frac{\|H \otimes I_n\|}{(\lambda_{\max}(E))^{1+c/2d}}V_1^{1+c/2d} \\
=& -\alpha_1 V_1^{1-c/2d} - \alpha_2 V_1^{1+c/2d}.
\end{aligned} \tag{23}$$

Using Lemma 1, ξ^* converges to zero in fixed time. Since matrix L_1 in (11) is positive definite and diagonally dominant. Thus $\zeta = 0 \ \forall t \geq T_{m1}$ i.e., MASs network (7) converges to the equilibrium in fixed time

$$T_{m1} = \frac{\pi d}{c\sqrt{\alpha_1 \alpha_2}}. \tag{24}$$

Next, we show the fixed-time convergence of $\tilde{z}_1$ at the equilibrium. Consider a Lyapunov candidate $V_2 = \frac{1}{2}(\xi_1)^\top (E \otimes I_n)\xi_1$. The derivative of V_2 is obtained as

$$\dot{V}_2 = (\xi_1)^\top (E \otimes I_n)\dot{\xi}_1 = \frac{1}{2}(\xi_1)^\top (H \otimes I_n) z_2^*. \tag{25}$$

Using (12), we have two cases:

Case 1: when $\underline{|\xi_{1,i}| > \varepsilon}$,

$$\begin{aligned}\dot{V}_2 =& \frac{1}{2}(\xi_1)^\top (H \otimes I_n)\left(\rho_3 (\xi_1)^{1-c/d} - \rho_4 (\xi_1)^{1+c/d}\right) \\ =& -\frac{\rho_3}{2}(\xi_1)^\top (H \otimes I_n)(H \otimes I_n)^{-1+c/d}((H \otimes I_n)\xi_1)^{1-c/d} \\ & -\frac{\rho_4}{2}(\xi_1)^\top (H \otimes I_n)(H \otimes I_n)^{-1-c/d}((H \otimes I_n)\xi_1)^{1+c/d}.\end{aligned} \tag{26}$$

Following similar steps used for convergence of V_1, we get

$$\begin{aligned}\dot{V}_2 =& -\frac{\rho_3}{2}\|(H \otimes I_n)\|_2^{-1+c/d}\|(H \otimes I_n)\xi_1\|_2^{2-c/d} \\ & -\frac{\rho_4}{2}\|(H \otimes I_n)\|_2^{-1-c/d}((nN)^{-c/2d}\|(H \otimes I_n)\xi_1\|_2^{2+c/d} \\ =& -\alpha_3 V_2^{1-c/2d} - \alpha_4 V_2^{1+c/2d}.\end{aligned} \tag{27}$$

Using Lemma 1, $\tilde{z}_1 = \tilde{z}_2 = 0$ with a convergence time $T_{m2} = \frac{\pi d}{c\sqrt{\alpha_3 \alpha_4}}$. Thus the maximum time for consensus in velocity and position is $T_1 = T_{m1} + T_{m2} = \frac{\pi d}{c\sqrt{\alpha_1 \alpha_2}} + \frac{\pi d}{c\sqrt{\alpha_3 \alpha_4}}$.

Case 2: when $\underline{|\xi_{1,i}| < \varepsilon}$,

$$\begin{aligned}\dot{V}_2 =& \frac{1}{2}(\xi_1)^\top (H \otimes I_n)(-\beta \xi_1) \\ =& -\frac{\beta}{2}(\xi_1)^\top (H \otimes I_n)\xi_1 \\ \leq& -\frac{\beta}{2}\lambda_{min}(H)\|\xi_1\|_2^2 \\ \leq& -\beta \lambda_{min}(H) V_2.\end{aligned} \tag{28}$$

From the above inequality, exponential convergence of $\tilde{z}_1 = \tilde{z}_2 = 0$ can be achieved. ∎

V. Design of BF-ISM control

In this section, we design the robust control law based on ISM control. The ISM surface for the follower agent is designed as

$$s_i = \xi_{2,i} - \xi_{2,i}(0) - \int_0^t u_{nom,i} d\tau, \tag{29}$$

where $\xi_{2,i}(0)$ is the initial condition of the disagreement dynamics (7). The time derivative of (29) in the compact form can be written as

$$\dot{s} = (L_1 \otimes I_n)(u_d + \Delta), \tag{30}$$

where u_d is the robust control vector to handle disturbances. The robust control is designed as

$$u_d = -K \text{sgn}(s). \tag{31}$$

If the upper bound of disturbance is known, then selecting the gain K greater than the maximum bound, results in the confinement of the sliding variable in the following set

$$\Omega = \{z_i \in \mathbb{R}^2 : s_i = 0\}. \tag{32}$$

This is known as the sliding manifold.

As the upper bound of the disturbance is considered to be unknown, we proceed with the design of the robust control using a barrier function approach, which is motivated by [14]. We introduced the set

$$\mathscr{B}_s = \{z_i \in \mathbb{R}^2 : |s_i| \leq \vartheta\}, \tag{33}$$

where ϑ is a positive design constant, defining the size of the region into which s_i will converge. The robust control term for i^{th} follower is designed as

$$u_{d,i} = K_i(|s_i|)\text{sgn}(s_i), \tag{34}$$

with the barrier gain

$$K_i(|s_i|) = \frac{\bar{\delta}}{\vartheta - |s_i|}, \tag{35}$$

where $\bar{\delta} = \vartheta \delta$ with $\delta > 0$. The gain (35) is chosen such that $K_i(|s_i|) \to \infty$ as $|s_i| \to \vartheta$ and $K_i(|s_i|) = \delta$ at $|s_i| = 0$. Thus, δ is the minimum gain of the robust control.

Next, we verify the attractiveness of (33), i.e., the BF-based ISM control (34) ensures that s_i converges to $\mathscr{B}_s \quad \forall i$.

Theorem 2: Under assumptions 1, the MASs dynamics (7) with the sliding surface (29) and the FXT-BF-ISM control law (34) ensures that practical sliding motion is maintained in the set (33).

Proof: Setting $t = 0$ in the sliding manifold, we find that $s_i(0) = 0$ for each of the surfaces and the trajectory of the system starts inside Ω. Further, to determine the existence of the sliding motion, a Lyapunov candidate is selected as

$$V_3 = \frac{1}{2}s^\top (E \otimes I_n)s + \frac{1}{2}(K(|s|) - K(0))^\top (E \otimes I_n)(K(|s|) - K(0)),$$

where $s = [s_1, \cdots, s_N]^\top$, $K(|s|) = \text{diag}\{K_1(|s_1|), \cdots, K_N(|s_N|)\}$.

Taking the time derivative of V_3 and using relation (30) and (35) in subsequent steps, it can be written as

$$\begin{aligned}\dot{V}_3 =& s^\top (E \otimes I_n)\dot{s} + (K(|s|) - K(0))^\top (E \otimes I_n)\dot{K}(|s|) \\ =& s^\top ((EL_1 + L_1^\top E) \otimes I_n)(-K(|s|)\,\text{sgn}(s) + \Delta) \\ & + (K(|s|) - K(0))^\top (E \otimes I_n)\mathbf{1}_N \left(\frac{\bar{\delta}}{(\vartheta - |s_i|)^2}\right)\text{sgn}(s)\dot{s} \\ =& s^\top (H \otimes I_n)(-K(|s|)\,\text{sgn}(s) + \Delta) + (K(|s|) - K(0))^\top \\ & \times ((H \otimes I_n)\mathbf{1}_N \phi(-K(|s|)\,\text{sgn}(s) + \Delta) \\ \leq& -(K_i(|s_i|) - \Delta_m)\|H \otimes I_n\|\|s\| - (K_i(|s_i|) - \Delta_m)\phi\|H \otimes I_n\| \\ & \times \|K(|s|) - K(0)\|,\end{aligned} \tag{36}$$

where $\frac{\bar{\delta}}{(\vartheta-|s_i|)^2} = \phi > 0$.
We take up two conditions. First, it may be such that the minimum gain is sufficient to reject the disturbances. The next case is where the minimum gain is lesser than the maximum bound of disturbance, where the barrier function comes into play.

- **Condition 1:** Suppose $\delta > \Delta_m$, then the last expression can be rewritten as

$$\begin{aligned}\dot{V}_3 &\leq -(\delta-\Delta_m)\lambda_{min}(H)\|s\| \\ &\quad -(\delta-\Delta_m)\phi\lambda_{min}(H)\|K(|s|)-K(0)\| \\ &\leq -\eta_1\lambda_{min}(H)\|s\| - \eta_1\lambda_{min}(H)\phi\|K(|s|)-K(0)\|,\end{aligned} \tag{37}$$

  where $\delta-\Delta_m-\eta_1 > 0$. This implies that $s_i(0) = 0$ and the sliding motion is maintained in Ω.
- **Condition 2:** If $\delta < \Delta_m$, then for $\lambda \in (0,1)$ with $\bar{\delta} = \delta\vartheta > 0$ such that $K(\lambda\vartheta) > \Delta_m$. Based on the above statement, we introduce the set

$$\bar{\mathscr{B}}_s = \{z_i \in \mathbb{R}^2 : |s_i| \leq \bar{\vartheta}\}, \tag{38}$$

  where $\bar{\vartheta} = \lambda\vartheta$. This means that outside $\bar{\mathscr{B}}_s$, the condition (37) is satisfied which means $s_i \notin \bar{\mathscr{B}}_s$. This implies that the set $\bar{\mathscr{B}}_s$ is positively invariant.

From the above analysis, it can be inferred that the system trajectory is always confined $s_i \in \mathscr{B}_s$, as $\Omega \subset \bar{\mathscr{B}}_s \subset \mathscr{B}_s$. This completes the proof.

■

VI. Simulation Results

In this section, a numerical simulation is performed to verify the theoretical results and to show the effectiveness of the proposed controller. The simulation is done for a leader-follower MASs network with 4 followers denoted by $F_i, i = 1,2,3,4$ and a single leader denoted by L as depicted by the directed communication topology $\mathscr{G}_d$ in Fig. 1. The Laplacian and pinning matrices of the digraph are given below.

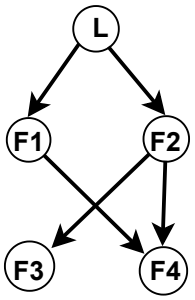


Fig. 1. Digraph $\mathscr{G}_d$

$$\mathscr{L}_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

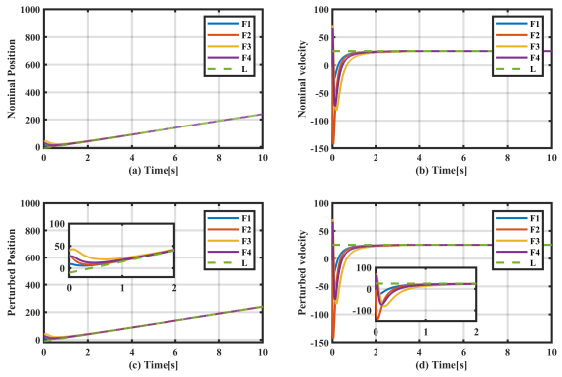


Fig. 2. (a) Nominal position consensus (b) Nominal velocity consensus (c) Perturbed position consensus (d) Perturbed velocity consensus

For the double integrator dynamics of MASs (5a) and (6a), the initial states of the leader are set as $z_{l,1}(0) = -10, z_{l,2}(0) = 25$, and the initial states of the followers are set as $z_1(0) = (-15,-5,5,15)^\top$, $z_2(0) = (-15,-5,5,15)^\top$. The parameters of the FXT protocol (14) are chosen as $\rho_1 = \rho_2 = 1$, $\rho_3 = \rho_4 = 1$, $\beta = 4.23$, $c/d = 3/5$ and $\varepsilon = 0.1$. The disturbance Δ for follower agents are considered as $\begin{bmatrix}0.2\mathbf{cos}(10t) & 0.5\mathbf{sin}(5t) & 0.2\mathbf{cos}(12t) & 0.5\mathbf{sin}(15t)\end{bmatrix}$. The parameters of the adaptive gain of the robust control (35) are selected as $\vartheta = 0.15$, $\delta = 0.3$. The response plot for leader-follower MASs with the proposed FXT-BF-ISM protocol is shown in Fig. 2 and 3.

The position consensus results of all follower agents are shown in Figs. 2(a) and 2(c). All followers track the leader by the fixed settling time $T_1 = 2$ sec. The leader's trajectory is shown by a dashed line. The velocity consensus trajectories of all follower agents are shown in Figs. 2(b) and 2(d). The convergence is obtained by the time $T_{m2} = 2$ sec.

From the disagreement plot shown in 3(a), it can be observed that position and velocity disagreement converge to zero around 3 sec. The plot of the adaptive gain is shown in 3 (b). The plot of the integral sliding variable s is depicted in Fig. 3 (c), where each sliding variable starts from zero at $t = 0$, evolves, and remains confined within the PSM bound of ϑ. The chattering in the response curve is due to the discontinuous sign function used in the ISM robust control. The BF-ISM robust control plot is shown in Fig. 3(d). The response obtained is reasonable, with the tracking error converging to the origin within a fixed time. It is important to note that the practical stability of the system has been compromised in order to relax prior assumptions on the disturbance bound and to address the issue of adaptive gain overestimation.

VII. Conclusion

This study designs and discusses the FXT-BF-ISM consensus mechanism for leader-follower problems of perturbed double integrator agents. A barrier function-based ISM protocol is proposed to mitigate disturbances and ensure a

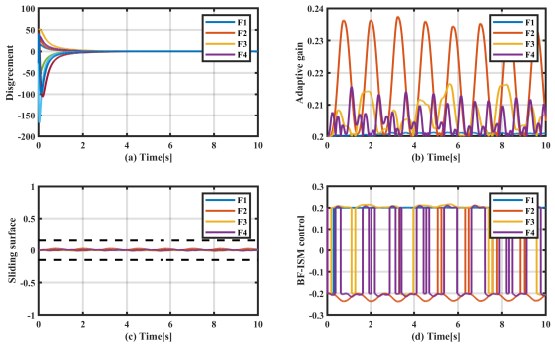


Fig. 3. (a) disagreement (b) Adaptive gain (c) Sliding surface (d) BF-ISM control of MASs

resilient response of MASs without requiring knowledge of the disturbance's upper bound. The suggested protocol achieves practical sliding motion. Theoretical contributions are illustrated through a numerical example. Future research will focus on extending the design to MASs having general linear dynamics. Additionally, the current effort is based on time-based sampling, and the next step will be to explore the event-triggered design of the FXT technique.

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