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On the Average Controllability Centrality of Laplacian Dynamics

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Abstract— **In this paper, the Laplacian network dynamics is considered, in which every node follows the classical *nearest-neighhbour* rule. One of the nodes, referred as the *control node*, is allowed to receive the external control input. To quantify the control energy required for the state transfer, the *average controllability centrality (ACC)* is prevalently used in the literature. The ACC of a node is defined as the trace of the *Controllability Grammian (CG)* matrix of the network dynamics when that node is acting as the control node. In this paper, the effect of network topology and control horizon, on the ACC is investigated. Subsequently, it is shown that under either of the following conditions, the ACC's of all nodes are almost equal:**
i) Networks with high edge density
ii) Too short and too long control horizon
As a result, it is concluded that under these conditions, the ACC is not an efficient metric for selecting the energy-efficient control node. Finally, the proposed results are demonstrated through simulations on randomly generated graphs.

Index Terms— **Average Controllability, Laplacian Dynamics**

I. Introduction

The controllability of a dynamical system refers to the ability of transferring the system trajectory from one state to another [1]. Over the last few years, the controllability of dynamical networks has received significant interest [2]-[4]. The main research problem in this area is to find the best nodes from which the network can be controlled efficiently. This problem has various applications such as leader selection in multi-agent systems [11], actuator placement in networked systems [12], finding the nodes most vulnerable to cyber-physical attack [13]. In order to quantify the required control efforts, a metric named as the *average controllability (AC)* has been prevalently used in the literature [15], [17], [18] [24], mainly due to its computational advantage. In this paper, we investigate the effect of the network topology and the control horizon, on the AC. Subsequently, we characterize the conditions under which the AC is not an effective metric for selecting nodes.

The notion of controllability was first introduced by Kalman in [1]. To test the controllability of a linear time-invariant (LTI) system, two rank conditions are widely used: the Kalman condition [1] and the Popov-Belevitch-Hautus condition [5]. These rank conditions are binary in nature. They just determine whether the given system is controllable or not. They do not quantify the control effort required for the state transfer. It is possible that a system is controllable according to the above rank conditions, but a certain state transfer requires control input with practically infeasible energy. In fact, an example of such a dynamical network is presented in [4].

In order to quantify the control energy required for the state transfer, various energy-based metrics have been proposed in the literature. These metrics are defined in terms of the eigenvalues of a special matrix called as the *controllability grammian (CG)* [5]. This matrix is a function of the state matrix, the input matrix and the control horizon of state transfer. The reason behind defining controllability metrics in terms of the CG matrix is the well-known fact that the minimum control energy required for transferring the system trajectory from one state to another is a function of the inverse of the CG matrix [5].

As the focus of this paper is on linear dynamical networks, we next discuss various energy-based controllability metrics proposed for them. In [6], [7], [8] and [9], the minimum eigenvalue of the CG matrix is used as a controllability metric. In [7], [10], and [11], the determinant of the CG matrix is employed as a controllability metric. The trace of the inverse of the CG matrix is studied in [11], [12] and [14]. These metrics have one major issue. They are computationally challenging and the optimization problems involving them are intractable for large networks [15]. To avoid this issue, many researchers have used the trace of the CG matrix as a controllability metric, due to its computational advantage. This metric is known as the *average controllability (AC)* [4].

For a linear dynamical network, the expression of AC in terms of the network matrix is obtained in [4]. In [11], the expression of AC in terms of the eigenstructure of the network matrix, is obtained. In [16], the lower and upper bounds on AC are computed. The problem of maximizing the AC under the constraint on the number of control nodes, is investigated in [17], [18], [19] and [20]. The data-driven techniques to measure and maximize the AC of uncertain networked systems are developed in [21] and [22].

In dynamical networks, to select the best nodes, it is necessary to know the relative importance of the individual nodes with respect to the given metric. Such relative importance is known as the *centrality measure* [23]. The centrality measure corresponding to the AC was defined and studied in [15]. It is called as the *average controllability centrality (ACC)*. Later, [24] obtained the expression for the ACC in terms of the eigenstructure of the Laplacian matrix.

Note that the above-mentioned works on AC/ACC have two shortcomings. First, they studied the AC/ACC only for the infinite control horizon. They did not investigate the effect of varying control horizon on AC/ACC. Second, the relation between the AC/ACC and the underlying network topology, is not well unexplored. In this paper, we attempt to fill these gaps.

In this paper, we consider a dynamical network with

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undirected connected underlying graph. Every node in the network updates its state according to the classical *nearest-neighbour* rule [2]. The resulting network dynamics is known as the *Laplacian dynamics* and has been extensively used [16], [25] in various areas such as co-operative coordination, sensor fusion, distributed optimization, etc. In addition to this dynamics, we allow one of the nodes to be a *control node* at which the external control input can be applied. We use the (finite-horizon) AC as the controllability metric and investigate the effect of network topology and control horizon on the ACC.

Our contributions in this paper are as follows:
1) We obtain the closed-form expression for the ACC of every node in terms of the control horizon and the eigenstructure of the network's Laplacian matrix (Theorem 5).
2) We prove that for every connected graph and positive control horizon, the absolute difference between the ACC's of any two nodes is upper bounded by the inverse of the second smallest eigenvalue of the Laplacian matrix, which is known as the *algebraic connectivity* [26] (Theorem 6).
3) Consequently, we show that under either of the following conditions, the ACC's of all nodes are almost equal:

i) Networks with high edge density (Theorem 7)

ii) Too short or too long control horizon (Theorems 8, 9)

As a result, we conclude that under these conditions, the ACC is not an effective metric for selecting the most energy-efficient node.
4) Finally, we present the simulations on the randomly generated graphs to illustrate our results (Section VI).

The remaining paper is organized as follows. In Section II, the description of the dynamical network and the ACC, is presented. In Section III, the analytical expression of the ACC in terms of the control horizon and the eigenstructure of the network Laplacian, is obtained. The effect of network topology on the ACC is analyzed in Section IV. In Section V, the effect of control horizon on the ACC is investigated. The simulation results on the randomly generated graphs are presented in Section VI. The paper is concluded in Section VII with the future directions.

II. Preliminaries and problem formulation

A. Graphs [26]

A graph $G = (V, E)$ is a set of nodes V connected by a set of edges $E \subseteq (V \times V)$. A graph G is said to be *undirected* if $(i, j) \in E$ implies $(j, i) \in E$. In an undirected graph, if $(i, j) \in E$, then the nodes i and j are called as the *neighbours* of each other. A graph is said to be *simple* if $(i, i) \notin E,\ \forall i \in V$. A *path* between nodes i and j is a sequence of edges connecting them. An undirected graph is called as *connected* if there exists a path between any two nodes in it.

Let n be the number of nodes in V and k_i be the number of neighbours, i.e., the degree, of node i. Then, the *Laplacian* matrix $L \in \mathbb{R}^{n\times n}$ of a simple undirected graph $G = (V, E)$ is defined as

$$L_{i,j} := \begin{cases} k_i, & \text{if} \quad i = j \\ -1, & \text{if} \quad i \neq j \quad \text{and} \quad (i,j) \in E \\ 0, & \text{if} \quad i \neq j \quad \text{and} \quad (i,j) \notin E \end{cases} \tag{1}$$

The *edge density* $d\big(G\big)$ of an undirected simple graph $G = \big(V, E\big)$, is defined as the ratio of the number of edges in it to the number of maximum possible edges among vertices in V (which occurs in the complete graph), i.e.,

$$d\big(G\big) := \frac{2|E|}{|V||V-1|} \tag{2}$$

where $|.|$ denotes the cardinality of a set.

B. Network dynamics

Consider a network of n nodes with the underlying undirected connected simple graph $G = (V, E)$. Let S_i denote the set of indices of the neighbours of node i. Let $x_i(t) \in \mathbb{R}$ be the state [1] of node i at instant t.

We consider the classical nearest-neighbour rule [2], according to which the dynamics of nodes is

$$\dot{x}_i(t) = -\sum_{j\in S_i} \big(x_i(t) - x_j(t)\big) \qquad \forall i = 1, \ldots, n \tag{3}$$

Let $L \in \mathbb{R}^{n\times n}$ be the Laplacian matrix of the underlying graph G as defined in (1). Let $X(t) := [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ denote the augmented state of all nodes in the network. Then, it is easy to see that the dynamics (3) can be compactly written as

$$\dot{X}(t) = -LX(t) \tag{4}$$

This dynamics is known as the *Laplacian dynamics*. It has been used [16], [25] in various applications such as co-operative coordination of autonomous vehicles, sensor fusion, distributed optimization, etc.

In addition, to steer the augmented state of the network, it is necessary to apply the external control input at one or more nodes, which are referred to as the *control nodes*. As our objective is to investigate the average controllability centrality of individual nodes, we consider only one control node. Let node i be the control node and u_i be the control input applied at node i. Let $e_i \in \mathbb{R}^n$ denote the ith canonical unit vector and $u_i : [0, \infty) \to \mathbb{R}$ denote the control input applied at node i. Then, the network dynamics (4) becomes

$$\dot{X}(t) = -LX(t) + e_i u_i(t) \tag{5}$$

C. Energy-based controllability metrics

Consider the dynamic network (5). As the underlying network graph is connected, the matrix L is symmetric. Then, the Controllability Grammian matrix of (5) over the interval $[0, T]$, denoted by $W_{i,T}$, is an $n \times n$ matrix defined as

$$W_{i,T} := \int_0^T e^{-L\tau} e_i e_i^\intercal e^{-L\tau} d\tau \tag{6}$$

where $e_i^\intercal$ denotes the transpose of matrix e_i.

It is well-known [5] that $W_{i,T}$ is a *symmetric* and *positive semidefinite* matrix for every $T \geq 0$. This matrix is related to the control energy required for state transfer as follows.

Consider the control input $u_i(t)$ over the interval $[0, T]$. The energy of this control input is defined as

$$E\big(T, u_i\big) := \int_0^T \big|\big|u_i(t)\big|\big|^2 \, dt \tag{7}$$

[1]The results proposed in this paper can be easily extended to higher dimensional node state, i.e., $x_i(t) \in \mathbb{R}^k$, by using the Kronecker product.

Then, it is well-known [5] that the minimum-energy control which transfers the trajectory of network (5) from $X(0) = 0$ to $X(T) = X_f$ is

$$u_i^*(t) = B^\intercal e^{(T-t)} W_{i,T}^{-1} X_f, \qquad t \in [0, T] \tag{8}$$

and the corresponding minimum control energy is

$$E\big(T, u_i^*\big) = X_f^\intercal W_{i,T}^{-1} X_f \tag{9}$$

Note that the minimum energy $E\big(T, u_i^*\big)$ is related to the matrix $W_{i,T}$. Hence, various energy-based controllability metrics are defined in terms of the eigenvalues of $W_{i,T}$. Among these metrics, the trace of $W_{i,T}$, denoted by $Tr\big(W_{i,T}\big)$, is prevalently used in the literature due to its computational advantage. This metric is known as the *average controllability (AC)*.

D. Average Controllability Centrality

In the graph theory, the *centrality* of a node refers to its relative importance among all nodes with respect to the given metric. The centrality measure corresponding to the average controllability is defined as follows.

Definition 1: [15] Consider the dynamic network (5) in which node i is the only control node. Let $Tr\Big(W_{i,T}\Big)$ be the trace of the matrix $W_{i,T}$ defined in (6). Then, the *average controllability centrality (ACC)* of node i corresponding to control horizon T, denoted by $C\big(i, T\big)$, is defined as

$$C\big(i, T\big) := Tr\Big(W_{i,T}\Big) \tag{10}$$

In this paper, our objective is to investigate the effect of the control horizon T and the network topology on $C\big(i, T\big)$. We formalize our objective as follows.

Problem 2: Consider dynamic network (5) with the Laplacian matrix L. Let $C\big(i, T\big)$ be as defined in (10). Then,
i) For every node i, obtain the expression of $C\big(i, T\big)$ in terms of the eigenstructure of L.
ii) Analyze the relation between $C\big(i, T\big)$'s and the network topology.
iii) Analyze the effect of varying T on $C\big(i, T\big)$'s.

III. Analytical expression of $C\big(i, T\big)$

In this section, we first obtain the eigenstructure of the network's Laplacian matrix L and then express $C\big(i, T\big)$ in terms of that eigenstructure.

A. Eigenstructure of Laplacian L

Recall that the underlying graph G of network (5) is undirected and hence, its Laplacian matrix L is symmetric. Thus, the eigenvalues of L are real and it is diagonalizable with real, orthogonal eigenvectors [27, Chapter 8]. Recall also that the graph G is connected. Thus, L has a few special properties which we present in the next proposition.

Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be the eigenvalues of L in an increasing order, with the corresponding normalized eigenvectors $v_1, \ldots, v_n \in \mathbb{R}^n$. Define the diagonal matrix

$$D := diag\ \big\{\lambda_1, \lambda_2, \ldots, \lambda_n\big\} \in \mathbb{R}^{n \times n} \tag{11}$$

and the eigenvector matrix

$$Q := [v_1, \ldots, v_n] \in \mathbb{R}^{n \times n} \tag{12}$$

Let I be the $n \times n$ identity matrix. Then, the eigenstructure of L has the following properties:

Proposition 3: [27, Chapter 8][26, Chapter 4] Let L be the Laplacian matrix of a connected undirected graph G. Let the eigenpairs (λ_j, v_j) of L and the matrices D, Q and I be as defined above. Then, the following holds.
1) $\lambda_1 = 0$ and $v_1 = \big[\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\big]^\intercal \in \mathbb{R}^n$.
2) $\lambda_j > 0$ for $j = 2, \ldots, n$.
3) L is diagonalizable as $L = QDQ^{-1}$.
4) Q is an orthogonal matrix, i.e., $Q^\intercal Q = I$.

B. Expression of $C\big(i, T\big)$

Consider the eigenstructure of L given in Proposition 3. Let $e_i \in \mathbb{R}^n$ be the ith canonical unit vector. Define

$$b_i := [b_{i_1}, b_{i_2}, \ldots, b_{i_n}]^\intercal = Q^{-1} e_i \tag{13}$$

Now, we prove some important properties of b_i, which will be used later.

Lemma 4: Let b_i be as defined in (13). Then, for every node i, the following holds.

1) $b_{i_1} = \dfrac{1}{\sqrt{n}}$ 2) $\sum_{j=1}^n b_{i_j}^2 = 1$

Proof: Recall from (12) that $Q = [v_1, \ldots, v_n]$, where v_i's are the eigenvectors of Laplacian L. Then, it follows from (13) that

$$e_i = Q b_i = \sum_{j=1}^{n} b_{i_j} v_j \tag{14}$$

We know from Proposition 3.4 that the vectors v_j's are orthonormal, i.e.,

$$v_j^\intercal v_j = 1 \quad \text{and} \quad v_j^\intercal v_l = 0 \text{ if } j \neq l \tag{15}$$

Then, (14) leads to $e_i^\intercal v_1 = \sum_{j=1}^n b_{i_j} v_j^\intercal v_1 = b_{i_1}$. Subsequently, it follows from Proposition 3.1 that $b_{i_1} = \dfrac{1}{\sqrt{n}}$. This proves the first claim.

Next, it follows from (14) that

$$e_i^\intercal e_i = ||e_i||_2^2 = 1 = \sum_{j=1}^{n} \sum_{l=1}^{n} b_{i_j} b_{i_l} v_j^\intercal v_l$$

Then, (15) leads to $\sum_{j=1}^n b_{i_j}^2 = 1$. This proves the second claim and completes the proof. ∎

Now, by using Lemma 4, we obtain the expression of $C\big(i, T\big)$ in the following theorem.

Theorem 5: Consider the dynamic network (5) with n nodes and the Laplacian matrix L. Let λ_i's and b_i's be as in (11) and (13), respectively. Then, the expression of $C\big(i, T\big)$ defined in (10) is

$$C\big(i, T\big) = \frac{T}{n} + \sum_{j=2}^{n} \frac{b_{i_j}^2 \big(1 - e^{-2\lambda_j T}\big)}{2\lambda_j} \tag{16}$$

Proof: Recall from Proposition 3.3 that $L = QDQ^{-1}$. Then, it follows from the definition of matrix exponential [28, Chapter I] that $e^{-L} = Qe^{-D}Q^{-1}$. Then, (13) leads to $e^{-L\tau} e_i = Qe^{-D\tau} Q^{-1} e_i = Qe^{-D\tau} b_i$.

Now, consider the expression of $W_{i,T}$ in (6). By putting the expression of $e^{-L\tau} e_i$ in (6), we get

$$W_{i,T} = \int_0^T Q e^{-D\tau} b_i b_i^\intercal e^{-D\tau} Q^\intercal d\tau$$

After interchanging the order of integration and multiplication by matrix Q, we get

$$W_{i,T} = Q\left(\int_0^T \ e^{-D\tau} b_i b_i^\intercal e^{-D\tau} d\tau \right) Q^\intercal$$

According to the associative property of the trace [27], $Tr\Big(W_{i,T}\Big)$ is equal to

$$Tr\Big(W_{i,T}\Big) = Tr\left(Q^\intercal Q \int_0^T \ e^{-D\tau} b_i b_i^\intercal e^{-D\tau} d\tau \right)$$

Recall from Proposition 3.4 that $Q^\intercal Q = I$. Thus,

$$Tr\Big(W_{i,T}\Big) = Tr\left(\int_0^T \ e^{-D\tau} b_i b_i^\intercal e^{-D\tau} d\tau \right) \tag{17}$$

Now, let $Y := e^{-D\tau} b_i b_i^\intercal e^{-D\tau}$ denote the matrix under integration in (17). We know from (11) that $e^{-D\tau}$ is a diagonal matrix as $e^{-D\tau} = diag\big\{e^{-\lambda_1\tau}, e^{-\lambda_2\tau}, \ldots, e^{-\lambda_n\tau}\big\}$. Thus, $e^{-D\tau} b_i = \big[b_{i_1} e^{-\lambda_1\tau},\ b_{i_2} e^{-\lambda_2\tau},\ \ \ldots, b_{i_n} e^{-\lambda_n\tau}\big]^\intercal$. After substituting this expression of $e^{-D\tau} b_i$ in Y, we get

$$Y = \begin{bmatrix} b_{i_1}^2 e^{-2\lambda_1\tau} & \star & \ldots & \star \\ \star & b_{i_2}^2 e^{-2\lambda_2\tau} & \ldots & \star \\ \vdots & \vdots & \ldots & \vdots \\ \star & \star & \ldots & b_{i_n}^2 e^{-2\lambda_n\tau} \end{bmatrix} \tag{18}$$

Next, we integrate the diagonal elements of Y over the interval $\big[0, T\big]$. Recall from Proposition 3.1 that $\lambda_1 = 0$. Recall also from Lemm 4.1 that $b_{i_1} = \frac{1}{\sqrt{n}}$. By using these facts, it is easy to show that

$$\int_0^T b_{i_1}^2 e^{-2\lambda_1\tau} d\tau = \frac{T}{n} \tag{19}$$

Further, recall from Proposition 3.2 that $\lambda_j > 0$ for $j = 2, \ldots, n$. Then, for each $j = 2, \ldots, n$, it is easy to show that

$$\int_0^T b_{i_j}^2 e^{-2\lambda_j\tau} d\tau = \frac{b_{i_j}^2 \big(1 - e^{-2\lambda_j T}\big)}{2\lambda_j} \tag{20}$$

Now, it follows from (17), (19) and (20) that

$$Tr\Big(W_{i,T}\Big) = \frac{T}{n} + \sum_{j=2}^{n} \ \frac{b_{i_j}^2 \big(1 - e^{-2\lambda_j T}\big)}{2\lambda_j} \tag{21}$$

Then, the definition of $C\big(i,T\big)$ in (10) leads to the claim (16) and completes the proof. ■

In this section, we obtained the expression $C\big(i,T\big)$ in terms of the eigenstructure of the network's Laplacian L. By using this expression, in the next section, we investigate the effect of network topology on $C\big(i,T\big)$'s.

IV. Effect of network topology on $C\big(i,T\big)$'s

In this section, we first obtain the bounds on $C\big(i,T\big)$ in terms of the control horizon T, the number of nodes n and the second smallest eigenvalue, i.e., the algebraic connectivity, of L, as follows.

Theorem 6: Consider the dynamic network (5) with n nodes and the Laplacian matrix L. Let λ_2 be the second smallest eigenvalue of L. Then, for every two nodes i and j and every positive control horizon T, we have:

$$\big|C\big(i,T\big) - C\big(j,T\big)\big| \leq \frac{1}{\lambda_2} \tag{22}$$

Proof: Consider the second term in the expression of $C\big(i,T\big)$ in (16). Recall from proposition 3.2 that $\lambda_j > 0$ for $j = 2, \ldots, n$. Then, for every $T > 0$, we have $0 \leq e^{-2\lambda_j T} < 1$. After simple algebraic manipulation, we get

$$0 < \frac{\big(1 - e^{-2\lambda_j T}\big)}{2\lambda_j} \leq \frac{1}{2\lambda_j}, \qquad \forall j = 2, \ldots, n \tag{23}$$

Recall that $\lambda_2 \leq \lambda_j$ for $j = 3, \ldots, n$. Then, (23) leads to

$$0 < \frac{\big(1 - e^{-2\lambda_j T}\big)}{2\lambda_j} \leq \frac{1}{2\lambda_2}, \qquad \forall j = 2, \ldots, n \tag{24}$$

Now, by multiplying equation (24) with $b_{i_j}^2$ and then taking summation, we get

$$0 < \sum_{j=2}^{n} \frac{b_{i_j}^2 \big(1 - e^{-2\lambda_j T}\big)}{2\lambda_j} \leq \frac{1}{2\lambda_2} \sum_{j=2}^{n} b_{i_j}^2 \tag{25}$$

Recall from Lemma 4.2 that $\sum_{j=1}^{n} b_{i_j}^2 = 1$ and hence, $\sum_{j=2}^{n} b_{i_j}^2 \leq 1$. Then, (25) leads to

$$0 < \sum_{j=2}^{n} \frac{b_{i_j}^2 \big(1 - e^{-2\lambda_j T}\big)}{2\lambda_j} \leq \frac{1}{\lambda_2} \tag{26}$$

After adding $\frac{T}{n}$ on both sides of (26) and comparing with (16), we get

$$\frac{T}{n} \leq C\big(i,T\big) \leq \frac{T}{n} + \frac{1}{\lambda_2} \qquad \forall i = 1, \ldots, n \tag{27}$$

Then, the claim (22) is straightforward. ■

Next, we use the bounds in (22) to investigate the effect of network topology on the distribution of $C\big(i,T\big)$'s. It is well-known [26] in the graph theory that the algebraic connectivity of the Laplacian matrix is positively correlated to the edge density $d\big(G\big)$, defined in (2). In the next theorem, we show that for networks with high $d\big(G\big)$, irrespective of T, the $C\big(i,T\big)$'s of all nodes are almost equal.

Theorem 7: Consider the dynamic network (5) with the underlying connected graph G. If G has high edge density $d\big(G\big)$, then for every $T > 0$, we have,

$$C\big(i,T\big) \approx C\big(j,T\big) \qquad \forall i, j = 1, \ldots, n \tag{28}$$

Proof: Consider the equation (22). It is well-known [26] that if the graph G has high edge density $d\big(G\big)$, then its algebraic connectivity λ_2 is high and hence, $\dfrac{1}{\lambda_2}$ is negligibly small. Then, (22) leads to

$$\big|C\big(i,T\big) - C\big(j,T\big)\big| \approx 0 \qquad \forall i, j = 1, \ldots, n$$

This proves the claim (28) and completes the proof. ■

The consequence of Theorem 7 is that for networks with high edge density, ACC is not an effective metric for selecting the best control node. In the next section, we investigate the effect of control horizon T on $C\big(i,T\big)$'s.

V. Effect of control horizon T on $C(i,T)$'s

In this section, we first obtain the approximate expressions of $C(i,T)$ for small and large T. Then, we show that for small and large T's, the $C(i,T)$'s of all nodes are almost equal, irrespective of the network topology.

A. Expression of $C(i,T)$ for small T

In the following theorem, we obtain the approximate expression of $C(i,T)$ for small T.

Theorem 8: Consider the dynamic network (5) with n nodes and connected graph G. Let $C(i,T)$ be as defined in (10). Then, for small T, we have

$$C(i,T) \approx \frac{(n-1)T}{n}, \quad \forall i = 1,\ldots,n \tag{29}$$

Proof: Consider the expression of $C(i,T)$ in (16). For small T, the first term $\frac{T}{n}$ in (16) is negligibly small. In addition, for small T, the exponential term $e^{-2\lambda_j T}$ can be approximated as $e^{-2\lambda_j T} \approx (1-2\lambda_j T)$. Then, (16) leads to

$$C(i,T) \approx T\sum_{j=2}^{n} b_{i_j}^2 \tag{30}$$

Next, it follows from Lemma 4 that

$$\sum_{j=2}^{n} b_{i_j}^2 = \sum_{j=1}^{n} b_{i_j}^2 - b_{i_1}^2 = 1 - \frac{1}{n} = \frac{n-1}{n} \tag{31}$$

Then, (30) and (31) together prove the claim (29). ∎

B. Expression of $C(i,T)$ for large T

In this section, we obtain the approximate expression of $C(i,T)$ for large T. In order to quantify the largeness of T, we define a threshold on T as follows.

Let n be the number of nodes in network (5) and λ_2 be the algebraic connectivity of the Laplacian L. Define the threshold T_c as

$$T_c := \frac{n}{\lambda_2} \tag{32}$$

Then, the following theorem shows that for T much larger than T_c, the $C(i,T)$'s of all nodes are almost equal.

Theorem 9: Consider the dynamic network (5) with connected graph G. Let $C(i,T)$ and T_c be as defined in (10) and (32), respectively. Then, for $T \gg T_c$, we have

$$C(i,T) \approx C(j,T), \qquad \forall i,j = 1,\ldots,n \tag{33}$$

Proof: Consider the definition of T_c in (32). Then, $T \gg T_c$ implies that

$$\frac{T}{n} \gg \frac{1}{\lambda_2} \tag{34}$$

Now, notice the lower and upper bounds on $C(i,T)$ in (27). It follows from (34) that for $T \gg T_c$, the relative difference between the lower bound and the upper bound on $C(i,T)$ (i.e., $\frac{T}{n}$ and $\frac{T}{n}+\frac{1}{\lambda_2}$, respectively), with respect to the lower bound (and hence, with respect to the nominal value), is negligible. As a result, for $T \gg T_c$, we have

$$\big|C(i,T) - C(j,T)\big| \approx 0, \qquad \forall i,j = 1,\ldots,n$$

This proves the claim (33) and completes the proof. ∎

The implication of Theorems 8 and 9 is that for small and large T, the ACC is not an effective metric for selecting the best control node.

VI. Simulation results

In this section, we present simulations to demonstrate the proposed results about the effect of the control horizon T and the edge density on $C(i,T)$'s.

A. Effect of T on $C(i,T)$'s

Consider the dynamic network (5) with $n = 7$ and the Laplacian matrix

$$L = \begin{bmatrix} 3 & 0 & -1 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & -1 & 0 & 4 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 6 \end{bmatrix}$$

For this network, the values of $C(i,T)$'s for various T's are given in Table I. Notice that for $T = 0.005$ sec and $T = 0.01$ sec, the $C(i,T)$'s corresponding to all nodes are close to $\frac{(n-1)T}{n} = 0.0043$ and $\frac{(n-1)T}{n} = 0.0086$, respectively. These observations are as per our claim in Theorem 8 for small T.

Next, the second smallest eigenvalue of L is $\lambda_2 = 0.7236$. Then, $T_c = \frac{n}{\lambda_2} = 5.0652$ sec. Observe the values of $C(i,T)$'s for $T = 50T_c = 5.0652$ sec. The maximum value of the difference $\big|C(i,50T_c) - C(j,50T_c)\big|$ is 0.1646 $\big($between $C(3,50T_c)$ and $C(7,50T_c)\big)$ which is 0.45% of the minimum value of $C(i,50T_c)$'s (which occurs for $C(7,50T_c) = 36.2416$). Similar observation can be made about $C(i,T)$'s for $T = 100T_c$. These observations are as per our claim in Theorem 9 for $T \gg T_c$.

B. Effect of edge density on $C(i,T)$'s

Consider the dynamic network (5) with $n = 100$ nodes. Let the control horizon be $T = 5$ sec. We begin with the edge density $d(G) = 0.1$ and increase it in the step size of 0.05 till it reaches the maximum value $d(G) = 1$. For each of these edge densities, we do the following:

1) We randomly generate 50 connected graphs with $n = 100$.
2) For each of these graphs, we compute the $C(i,T)$'s of all nodes. Then, we compute the maximum value of the difference $\big|C(i,T) - C(j,T)\big|$. We denote it by C_{diff}.
3) Note that if all $C(i,T)$'s are small, then naturally, C_{diff} is also small. Hence, C_{diff} is meaningful only after normalization with the average value of $C(i,T)$.
4) Thus, we normalize c_{diff} by dividing it with the average of $C(i,T)$'s. We denote this normalized maximum difference by C_{norm}.

Finally, we plot C_{norm} versus $d(G)$ as shown in Figure 1. Notice that with the increase in $d(G)$, the C_{norm} decreases, and eventually becomes 0 for $d(G) = 1$. This implies that for high edge density, the $C(i,T)$'s of all nodes are almost equal. This inference is as per our claim in Theorem 7.

VII. Conclusion

In this paper, we considered the Laplacian network dynamics with one control node and its finite-horizon average controllability centrality (ACC). We obtained the closed-form

TABLE I

SIMULATION A: AVERAGE CONTROLLABILITY CENTRALITY $C(i,T)$ OF ALL NODES

Node i	1	2	3	4	5	6	7
$C(i,0.005)$	0.0049	0.0050	0.0050	0.0049	0.0050	0.0049	0.0049
$C(i,0.01)$	0.0097	0.0098	0.0098	0.0097	0.0098	0.0096	0.0094
$C(i,50\,T_c)$	36.3283	36.3950	36.4061	36.3283	36.4061	36.2839	36.2416
$C(i,100\,T_c)$	341.1501	340.9599	341.0881	341.2756	341.5161	340.9579	340.6196

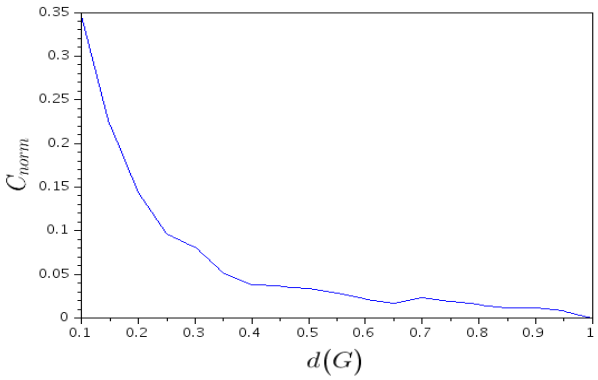


Fig. 1. Simulation B: C_{norm} versus $d(G)$

expression for ACC in terms of the control horizon and the eigenstructure of the network Laplacian. By using this expression, we proved that for every connected graph and every control horizon, the absolute difference between the ACC's of any two nodes is upper bounded by the inverse of the algebraic connectivity of the network Laplacian. Consequently, we showed that under either of the following conditions, the ACC's of all nodes are almost equal:

i) Networks with high edge density

ii) Too short or too long control horizon

This implies that the ACC is not an effective metric under these conditions. Finally, we demonstrated our results through simulations on the randomly generated graphs. The work of extending the proposed results to other controllability metrics is in progress.

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