2024 Tenth Indian Control Conference (ICC)
9 - 11 December, 2024
IISER Bhopal, India

Pareto optimality of stochastic mean-field delay systems

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Abstract— **This research focuses on identifying the necessary and sufficient conditions for achieving Pareto optimality in delayed cooperative differential games that are governed by mean-field stochastic differential equations over a finite horizon. We express the game problem as a set of constrained delayed mean-field stochastic optimal control problems. Through the application of the maximum principle for optimal control in mean-field stochastic delay differential systems, we derive the necessary condition for a Pareto efficient strategy. Additionally, we provide the sufficient condition under the convexity assumptions on certain functions. Finally, a linear quadratic optimal control problem is considered as an application of the theoretical findings.**

I. INTRODUCTION

Cooperative differential games (CDGs) arise when multiple players, influencing a dynamic system, coordinate their actions/controls with the aim of optimizing their objectives. Pareto optimality is essential for analyzing CDGs. Due to the presence of multiple players, no player can independently determine their cost. Each player encounters various possible outcomes based on how control efforts are distributed. When two control strategies, u and v, exist and all players have lower costs with strategy u, strategy u is considered to dominate strategy v. Therefore, dominance indicates that the outcome is improved for all players. When no other solution can dominate the one induced by strategy u^*, it is called a non-dominated or Pareto solution.

For deterministic systems, Pareto optimality has been thoroughly investigated by many authors (see [1]–[4]), whereas Pareto optimality in stochastic systems has become a focal point of research interest recently. The authors in [5] discussed the Pareto game for linear quadratic (LQ) stochastic singular systems in a finite horizon and derived all Pareto candidates using the solvability of generalized differential Riccati equations. In contrast, [6] derived all Pareto solutions for LQ stochastic singular systems in an infinite horizon by solving a Lyapunov equation. The necessary and sufficient conditions for the existence of the Pareto optimal strategy of the stochastic nonlinear systems in the finite horizon were established in [7].

In recent research, there has been growing interest in mean-field stochastic differential equations (MF-SDEs). Kac [8] pioneered the mean-field model to investigate physical systems containing numerous interacting particles. Since then, many authors have started examining problems involving differential games driven by MF-SDEs. Notably, extensive research has been carried out on the Pareto game related to MF-SDEs. To gain further understanding of the advancements related to MF-SDEs in the context of Pareto cooperative games, readers are encouraged to refer [9]–[11].

Many practical phenomena are characterized by past dependence, where behaviour at time t is influenced not only by the present situation but also by past history. Such models, known as stochastic delay differential equations (SDDEs), and are widely used in finance and economics. The study of optimal control problems (OCPs) for stochastic delayed systems and MF-SDEs with delay has been conducted in [12]–[17].

However, there are only a limited number of studies on the Pareto optimality of delayed systems [18]–[20]. It is evident that the referenced works on the Pareto optimality of delayed systems focus solely on stochastic systems without mean-field interactions. In reality, when dealing CDGs with numerous interacting players, the average of all players can significantly impact each individual player's payoff.

Inspired by this, we investigate the Pareto cooperative game for mean-field stochastic delay systems over a finite time horizon in this manuscript. The major contributions of this paper are as follows:

- The necessary condition for the existence of Pareto optimal control (POC) for a stochastic delay system of the mean-field type is derived using the maximum principle. This paper considers mean-field terms in a general formulation defined in product space (see [16], [21]).
- Subsequently, we establish sufficient conditions and demonstrate that these conditions are also necessary under the convexity assumptions of the Hamiltonian and the functions in the cost functional.
- The manuscript also addresses the application of the proposed theory. We discuss an LQ problem and obtain the optimal condition for the POC of the proposed linear game system.

The structure of the paper is as follows: Section 2 outlines the necessary and sufficient conditions for achieving POC of the game problem governed by MF-SDDEs. Section 3 delves into an LQ OCP and validates the findings from the non-linear case using a weighted cost functional. Finally, we conclude with a few remarks.

Notations: $\mathbb{R}^m$: the Euclidean space of dimension m. $\mathbb{S}^m$: the set of all $m \times m$ symmetric matrices. $M^\top$: the transpose of matrix M. $\mathcal{C} := \{\rho = (\rho_1, \cdots, \rho_M) |\ 0 < \rho_i <$

*This work was not supported by any organization

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1, and $\sum_{i=1}^{M} \rho_i = 1\}$. $\bar{M} := \{1, 2, \cdots M\}$. $\bar{M}\backslash i$: an element i is lacking in the set $\bar{M}$. $|\cdot|$, $\langle \cdot, \cdot \rangle$: the norm and inner product in the Euclidean space respectively.

The following are the spaces which are used in this manuscript.

- $\mathfrak{L}^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m) = \{\varrho : \varrho$ is an $\mathbb{R}^m-$valued $\mathcal{F}$ measurable random variable$\}$.
- For $p \geq 1$, $\mathfrak{L}^p(\Omega, \mathcal{F}_\mathfrak{s}, \mathbb{P}; \mathbb{R}^m) = \{\varrho : \varrho$ is an $\mathbb{R}^m-$valued $\mathcal{F}_\mathfrak{s}-$ measurable random variable; $\mathtt{E}[|\varrho|^p] < +\infty\}$.
- $\mathfrak{L}^p_\mathcal{F}(\mathfrak{t}_1, \mathfrak{t}_2; H) = \{H$ valued $\mathcal{F}_\mathfrak{s}-$adapted processes $\psi_\mathfrak{s}$, $\mathfrak{t}_1 \leq \mathfrak{s} \leq \mathfrak{t}_2$ such that $\mathtt{E}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2} |\psi_\mathfrak{s}|^p d\mathfrak{s} < +\infty\}$.
- $\mathfrak{L}^\infty(0, T; \mathbb{R}^m)$: the space of all essentially bounded functions from $[0, T]$ to $\mathbb{R}^m$.
- $\mathcal{S}^p_\mathcal{F}(0, T; \mathbb{R}^m) = \{\mathbb{R}^m-$ valued $\mathcal{F}_\mathfrak{s}-$adapted process $\psi_\mathfrak{s}$, $0 \leq \mathfrak{s} \leq T$, $\psi_\mathfrak{s}$ is right continuous with left limits : $\mathtt{E} \sup_{\mathfrak{s}\in[0,T]} |\psi_\mathfrak{s}|^p < +\infty\}$.

Let $T > 0$ and $0 < \tau \leq T$ denote the finite time horizon and a constant time delay, respectively. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_\mathfrak{s}\}_{\mathfrak{s}\geq 0})$ be a complete filtered probability space endowed with a natural filtration $\mathcal{F}_\mathfrak{s} = \sigma\{\mathcal{B}_\mathfrak{s} : 0 \leq \mathfrak{s} \leq \mathfrak{t}\}$, where $\{\mathcal{B}_\mathfrak{s}\}_{\mathfrak{s}\geq 0}$ is a one dimensional standard Brownian motion. Let us take $\mathcal{F}_\mathfrak{s} \equiv \mathcal{F}_0$ $\forall \mathfrak{s} \in [-\tau, 0]$, $\mathcal{F} = \mathcal{F}_{T+\tau}$, and denote by $\mathtt{E}$, the expectation on $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_\mathfrak{s}\}_{\mathfrak{s}\geq 0})$. Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) = (\Omega \times \Omega, \mathcal{F} \times \mathcal{F}, \mathbb{P} \times \mathbb{P})$ be the product space of $(\Omega, \mathcal{F}, \mathbb{P})$ with itself. The filtration on the product space is given by $\bar{\mathbb{F}} = \{\bar{\mathcal{F}}_\mathfrak{s} = \mathcal{F} \otimes \mathcal{F}_\mathfrak{s}, \ 0 \leq \mathfrak{s} \leq T\}$. A random variable $\varrho \in \mathfrak{L}^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$ can be extended to $\bar{\Omega}$ as $\varrho'(\omega', \omega) = \varrho(\omega')$, where $(\omega', \omega) \in \bar{\Omega} = \Omega \times \Omega$. For any $\varrho \in \mathfrak{L}^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, $\varrho(\cdot, \omega) : \Omega \longrightarrow \mathbb{R}$ belongs to $\mathfrak{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, almost surely with respect to $\mathbb{P}(d\omega)$. $\mathtt{E}'[\varrho(\cdot, \omega)] = \int_\Omega \varrho(\omega', \omega)\mathbb{P}(d\omega')$ represents its expectation. Observe that $\bar{\mathtt{E}}'[\varrho] = \mathtt{E}'[\varrho(\cdot, \omega)] \in \mathfrak{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. $\bar{\mathtt{E}}[\varrho] = \left(\int_{\bar{\Omega}} \varrho d\bar{\mathbb{P}} = \int_\Omega \mathtt{E}'[\varrho(\cdot, \omega)]\mathbb{P}d(\omega)\right) = \mathtt{E}[\mathtt{E}'[\varrho]]$.

II. Necessary and Sufficient Conditions for Pareto Optimality

This section examines a particular type of Pareto cooperative game for the following controlled mean-field stochastic system with delays in both state and control variables:

$$\begin{aligned} dx_\mathfrak{s} =& \mathtt{E}'\Big[f(\mathfrak{s}, x_\mathfrak{s}, x'_\mathfrak{s}, x_{\tau\mathfrak{s}}, x'_{\tau\mathfrak{s}}, v_{1\mathfrak{s}}, v_{1\tau\mathfrak{s}}, \cdots, v_{M\mathfrak{s}}, v_{M\tau\mathfrak{s}})\Big] d\mathfrak{s} \\ &+ \mathtt{E}'\Big[\sigma(\mathfrak{s}, x_\mathfrak{s}, x'_\mathfrak{s}, x_{\tau\mathfrak{s}}, x'_{\tau\mathfrak{s}}, v_{1\mathfrak{s}}, v_{1\tau\mathfrak{s}}, \cdots, v_{M\mathfrak{s}}, v_{M\tau\mathfrak{s}})\Big] d\mathcal{B}_\mathfrak{s}, \\ x_\mathfrak{s} =& x_{0,\mathfrak{s}}, \quad v_\mathfrak{s} = v_{0,\mathfrak{s}}, \quad \mathfrak{s} \in [-\tau, 0], \end{aligned} \tag{1}$$

where $x_\mathfrak{s} \in \mathbb{R}^m$ is the state vector, $x_{\tau\mathfrak{s}} \in \mathbb{R}^m$ is the state vector with delay τ, the control strategy of the i^{th} player without and with delay are denoted by $v_{i\mathfrak{s}}, v_{i\tau\mathfrak{s}}$ respectively, and they belong to $\mathfrak{L}^2_\mathcal{F}(0, T; U_i)$, where U_i is a nonempty convex subset of $\mathbb{R}^{n_i}$, for $i \in \bar{M}$ and $f, \sigma : [0, T] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times U_1 \times U_1 \times \cdots \times U_M \times U_M \longrightarrow \mathbb{R}^m$. $x_{0,\mathfrak{s}}, v_{0,\mathfrak{s}}$ are the initial paths of x and v respectively.

For each $i \in \bar{M}$, the cost functional associated with the player i is described as follows:

$$\begin{aligned} \mathcal{J}_i(v_1, \cdots, v_M) = \mathtt{E}\Bigg\{ \int_0^T & \mathtt{E}'[l_i(\mathfrak{s}, x_\mathfrak{s}, x'_\mathfrak{s}, v_{1\mathfrak{s}}, v_{1\tau\mathfrak{s}}, \cdots, v_{M\mathfrak{s}}, v_{M\tau\mathfrak{s}})] \\ & \times d\mathfrak{s} + \mathtt{E}'[k_i(x_T, x'_T)] \Bigg\}, \end{aligned} \tag{2}$$

where $l_i : [0, T] \times \mathbb{R}^m \times \mathbb{R}^m \times U_1 \times U_1 \times \cdots \times U_M \times U_M \longrightarrow \mathbb{R}$, $k_i : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$, $i \in \bar{M}$.

To ensure the well-posedness of the above problem, we introduce the following assumptions:

(H1) For any $x, x', x_\tau, x'_\tau \in \mathbb{R}^m$, $v_i, v_{i\tau} \in U_i$, $f(\cdot, x, x_\tau, x', x'_\tau, v_1, v_{1\tau}, \cdots, v_M, v_{M\tau})$, $\sigma(\cdot, x, x_\tau, x', x'_\tau, v_1, v_{1\tau}, \cdots, v_M, v_{M\tau})$, $i \in \bar{M}$ are adapted to $\{\mathcal{F}_\mathfrak{s}\}_{\mathfrak{s}\geq 0}$; for any $x, x' \in \mathbb{R}^m$, the terminal cost functions $k_i(x, x')$, $i \in \bar{M}$ are $\mathcal{F}_T-$measurable;

(H2) $f(\cdot, 0, 0, 0, 0, \cdots, 0, 0) \in \mathfrak{L}^2_\mathcal{F}(0, T; \mathbb{R}^m)$, $\sigma(\cdot, 0, 0, 0, 0, \cdots, 0, 0) \in \mathfrak{L}^2_\mathcal{F}(0, T; \mathbb{R}^m)$, $l_i(\cdot, 0, 0, 0, 0, \cdots, 0, 0) \in \mathfrak{L}^1_\mathcal{F}(0, T; \mathbb{R})$ and $k_i(0, 0) \in \mathfrak{L}^1(\bar{\Omega}, \bar{\mathcal{F}}_T, \bar{\mathbb{P}}; \mathbb{R})$, $i \in \bar{M}$;

(H3) f, σ and l_i, $i \in \bar{M}$ are continuously differentiable with respect to $x, x_\tau, x', x'_\tau, v_i, v_{i\tau}$, $i \in \bar{M}$ and $x, x', v_i, v_{i\tau}$, $i \in \bar{M}$ respectively. They and all their partial derivatives are uniformly bounded. k_i is continuously differentiable with respect to x, x' and its partial derivatives are bounded for $i \in \bar{M}$.

As the players are coordinating their actions, let us denote their joint action by $v_\mathfrak{s} := (v_{1\mathfrak{s}}^\top, v_{2\mathfrak{s}}^\top, \cdots, v_{M\mathfrak{s}}^\top)^\top \in \mathbb{R}^n$ with $n = \sum_{i=1}^M n_i$ and $U = U_1 \times \cdots \times U_M$. The set of all admissible controls is denoted by $\mathcal{U} = \mathfrak{L}^2(0, T; U_1) \times \cdots \times \mathfrak{L}^2(0, T; U_M)$. As we are focused on the joint minimization of all players' objective functions, the cost for one player cannot be minimized without increasing the cost for others. Thus, we consider solutions that cannot be improved upon by all players simultaneously, known as Pareto efficient strategies.

Definition 2.1 ([3]): $u \in \mathcal{U}$ is called Pareto efficient if the set of inequalities $\mathcal{J}_i(v) \leq \mathcal{J}_i(u)$, $i \in \bar{M}$, where at least one of the inequalities is strict, does not allow for any solution $v \in \mathcal{U}$. The corresponding point $(\mathcal{J}_1(u), \mathcal{J}_2(u), \cdots, \mathcal{J}_M(u)) \in \mathbb{R}^M$ is called a Pareto solution. The set of all Pareto solutions is termed as Pareto frontier.

The aim of this section is to determine the set of POCs for the finite horizon mean-field delayed stochastic CDG (1)-(2). The subsequent lemmas define Pareto efficient controls. Lemma 2.2 states that any control minimizing a weighted sum of all players' cost functions is Pareto efficient.

Lemma 2.2 ([22]): Suppose the admissible control set $\mathcal{U}$ and the cost functional $\mathcal{J}_i(v)$, $i \in \bar{M}$ are convex with respect to v. Then u^* is a Pareto optimal strategy, if and only if $u^* \in arg\min_{v\in\mathcal{U}} \left\{ \sum_{i=1}^M \rho_i \mathcal{J}_i(v) \right\}$, $\rho = (\rho_1, \rho_2, \cdots, \rho_M) \in \mathcal{C}$.

Lemma 2.3 ([1]): $u \in \mathcal{U}$ is Pareto efficient if and only if for all i, u minimizes $\mathcal{J}_i(u)$ on the constrained set $\mathcal{U}_i :=$

$\{v \in \mathcal{U} | \mathcal{J}_j(v) \leq \mathcal{J}_j(u), j = 1, 2, \cdots, M, j \neq i\}$, $i = 1, 2, \cdots, M$.

According to the above Lemma 2.3, all Pareto efficient strategies can be obtained by solving M constrained OCPs. Each of these problems, denoted as **(MF-P$_i$)** corresponds to a player $i \in \bar{M}$ and is formulated as follows: **(MF-P$_i$)** $\min_{u \in \mathcal{U}}$

$$\mathbb{E}\Big\{ \int_0^T \mathbb{E}'[l_i(\mathfrak{s}, x_\mathfrak{s}, x'_\mathfrak{s}, v_\mathfrak{s}, v_{\tau\mathfrak{s}})]d\mathfrak{s} + \mathbb{E}'[k_i(x_T, x'_T)] \Big\} \text{ such that}$$

$$\begin{aligned} dx_\mathfrak{s} &= \mathbb{E}'[f(\mathfrak{s}, x_\mathfrak{s}, x_{\tau\mathfrak{s}}, x'_\mathfrak{s}, x'_{\tau\mathfrak{s}}, v_\mathfrak{s}, v_{\tau\mathfrak{s}})]d\mathfrak{s} \\ &\quad + \mathbb{E}'[\sigma(\mathfrak{s}, x_\mathfrak{s}, x'_\mathfrak{s}, x'_{\tau\mathfrak{s}}, v_\mathfrak{s}, v_{\tau\mathfrak{s}})]d\mathcal{B}_\mathfrak{s}, \\ x_\mathfrak{s} &= x_{0,\mathfrak{s}}, \ v_\mathfrak{s} = v_{0,\mathfrak{s}}, \\ dy_{j\mathfrak{s}} &:= \mathbb{E}'[l_j(\mathfrak{s}, x_\mathfrak{s}, x'_\mathfrak{s}, v_\mathfrak{s}, v_{\tau\mathfrak{s}})]d\mathfrak{s}, \\ y_{j0} &= 0, \ \forall j \in \bar{M}\backslash i, \\ \mathbb{E}\Big[y_{jT} + \mathbb{E}'[k_j(x_T, x'_T)] \Big] &\leq c_j^\star, \ \forall j \in \bar{M}\backslash i. \end{aligned} \tag{3}$$

From the above analysis, the following lemma is obtained.

Lemma 2.4: u^* is a POC for the delayed mean-field CDG if and only if u^* is an optimal control for all constrained OCPs **(MF-P$_i$)**, $i \in \bar{M}$.

By employing the maximum principle for delayed MF-SDEs [16], we establish the necessary conditions for the Pareto optimal strategies in the next theorem.

*Theorem 2.5 (**Necessary conditions**):* Suppose u^* is a Pareto optimal strategy. Then $\exists\ \rho = (\rho_1, \rho_2, \cdots, \rho_M) \in \mathcal{C}$ and adjoint processes $(p^*, q^*) \in \mathcal{S}^2_\mathcal{F}(0, T; \mathbb{R}^m) \times \mathfrak{L}^2_\mathcal{F}(0, T; \mathbb{R}^m)$ such that u^* satisfies

$$\begin{aligned} &\mathbb{E}'\Big[\Big\langle \mathcal{H}_v(\mathfrak{s}, x^*_\mathfrak{s}, x^*_{\tau\mathfrak{s}}, x^{*'}_\mathfrak{s}, x^{*'}_{\tau\mathfrak{s}}, u^*, u^*_{\tau\mathfrak{s}}, p^*_\mathfrak{s}, q^*_\mathfrak{s}, \rho) \\ &+ \mathbb{E}^{\mathcal{F}_\mathfrak{s}}\Big[\mathcal{H}_{\tau\mathfrak{s}}(\mathfrak{s}, x^*_\mathfrak{s}, x^*_{\tau\mathfrak{s}}, x^{*'}_\mathfrak{s}, x^{*'}_{\tau\mathfrak{s}}, u^*, u^*_{\tau\mathfrak{s}}, p^*_\mathfrak{s}, q^*_\mathfrak{s}, \rho)|_{\mathfrak{s}+\tau} \Big], v - u^* \Big\rangle \Big] \\ &\geq 0, for\ all\ v \in U,\ a.e.\ \mathfrak{s} \in [0, T], \end{aligned} \tag{4}$$

where the Hamiltonian function $\mathcal{H}$ is of the form

$$\begin{aligned} &\mathcal{H}(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau, p, q, \rho) := \sum_{i=1}^M \rho_i l_i(\mathfrak{s}, x, x', v, v_\tau) \\ &+ p^\top f(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau) + q^\top \sigma(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau) \end{aligned} \tag{5}$$

and the adjoint process p^* satisfies the following mean-field anticipated backward stochastic differential equation:

$$\begin{aligned} dp^*_\mathfrak{s} &= -\Big\{ \mathbb{E}'\Big[\mathcal{H}_x(\mathfrak{s}, x^*_\mathfrak{s}, x^*_{\tau\mathfrak{s}}, x^{*'}_\mathfrak{s}, x^{*'}_{\tau\mathfrak{s}}, v^*_\mathfrak{s}, v^*_{\tau\mathfrak{s}}, p^*_\mathfrak{s}, q^*_\mathfrak{s}, \rho) \\ &\quad + \mathcal{H}_{x'}(\mathfrak{s}, x^*_\mathfrak{s}, x^*_{\tau\mathfrak{s}}, x^{*'}_\mathfrak{s}, x^{*'}_{\tau\mathfrak{s}}, v^*_\mathfrak{s}, v^*_{\tau\mathfrak{s}}, p^*_\mathfrak{s}, q^*_\mathfrak{s}, \rho) \\ &\quad + \mathbb{E}^{\mathcal{F}_\mathfrak{s}}[\mathcal{H}_{x_\tau}(\mathfrak{s}, x^*_\mathfrak{s}, x^*_{\tau\mathfrak{s}}, x^{*'}_\mathfrak{s}, x^{*'}_{\tau\mathfrak{s}}, v^*_\mathfrak{s}, v^*_{\tau\mathfrak{s}}, p^*_\mathfrak{s}, q^*_\mathfrak{s}, \rho)|_{\mathfrak{s}+\tau} \\ &\quad + \mathcal{H}_{x'_\tau}(\mathfrak{s}, x^*_\mathfrak{s}, x^*_{\tau\mathfrak{s}}, x^{*'}_\mathfrak{s}, x^{*'}_{\tau\mathfrak{s}}, v^*_\mathfrak{s}, v^*_{\tau\mathfrak{s}}, p^*_\mathfrak{s}, q^*_\mathfrak{s}, \rho)|_{\mathfrak{s}+\tau}] \Big] \Big\} \\ &\quad + q^*_\mathfrak{s} d\mathcal{B}_\mathfrak{s},\ \mathfrak{s} \in [0, T], \\ p^*_T &= \sum_{i=1}^M \rho_i \mathbb{E}'\Big[k_{ix}(x^*_T, x^{*'}_T) + k_{ix'}(x^*_T, x^{*'}_T) \Big], \\ p^*_\mathfrak{s} &= 0,\ \mathfrak{s} \in (T, T+\tau],\ q^*_\mathfrak{s} = 0,\ \mathfrak{s} \in [T, T+\tau]. \end{aligned} \tag{6}$$

Proof: Lemma 2.4 states that if u^* is a Pareto efficient strategy, then u^* also serves as the optimal control for all the minimization problems **(MF-P$_i$)**, $i \in \bar{M}$.

After combining the maximum principle stated in [16] and [18], we arrive at a conclusion that for each $i \in \bar{M}$, $\exists (p^*_i, q^*_i) \in \mathcal{S}^2_\mathcal{F}(0, T; \mathbb{R}^m) \times \mathfrak{L}^2_\mathcal{F}(0, T; \mathbb{R}^m)$ and $(\varphi^i_0, \varphi^i) \in \mathbb{R}^{1+\bar{M}\backslash i}$ such that, with the Hamiltonian

$$\begin{aligned} &\mathcal{H}^i(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau, p_i, q_i, \varphi^i_0, \varphi^i) := \varphi^i_0 l_i(\mathfrak{s}, x, x', v, v_\tau) \\ &+ \sum_{j \neq i} \varphi^i_j l_j(\mathfrak{s}, x, x', v, v_\tau) + \langle p_i, f(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau) \rangle \\ &+ \langle q_i, \sigma(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau) \rangle, \end{aligned} \tag{7}$$

the following holds

$$\begin{aligned} &\mathbb{E}'\Big[\Big\langle \mathcal{H}^i_v(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau, p^*_i, q^*_i, \varphi^i_0, \varphi^i) \\ &+ \mathbb{E}^{\mathcal{F}_\mathfrak{s}}[\mathcal{H}^i(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau, p^*_i, q^*_i, \varphi^i_0, \varphi^i)|_{\mathfrak{s}+\tau}], v - u^*_\mathfrak{s} \Big\rangle \Big] \\ &\geq 0, \forall v \in U a.e.\ \mathfrak{s} \in [0, T]\ \bar{\mathbb{P}} - a.s. \end{aligned} \tag{8}$$

where p^*_i, q^*_i satisfy the following anticipated MF-Backward SDE

$$\begin{aligned} dp^*_{i\mathfrak{s}} &= -\Big\{ \mathbb{E}'\Big[\mathcal{H}_x(\mathfrak{s}, x^*_\mathfrak{s}, x^*_{\tau\mathfrak{s}}, x^{*'}_\mathfrak{s}, x^{*'}_{\tau\mathfrak{s}}, v^*_\mathfrak{s}, v^*_{\tau\mathfrak{s}}, p^*_{i\mathfrak{s}}, q^*_{i\mathfrak{s}}, \rho) \\ &\quad + \mathcal{H}_{x'}(\mathfrak{s}, x^*_\mathfrak{s}, x^*_{\tau\mathfrak{s}}, x^{*'}_\mathfrak{s}, x^{*'}_{\tau\mathfrak{s}}, v^*_\mathfrak{s}, v^*_{\tau\mathfrak{s}}, p^*_{i\mathfrak{s}}, q^*_{i\mathfrak{s}}, \rho) \\ &\quad + \mathbb{E}^{\mathcal{F}_\mathfrak{s}}[\mathcal{H}_{x_\tau}(\mathfrak{s}, x^*_\mathfrak{s}, x^*_{\tau\mathfrak{s}}, x^{*'}_\mathfrak{s}, x^{*'}_{\tau\mathfrak{s}}, v^*_\mathfrak{s}, v^*_{\tau\mathfrak{s}}, p^*_{i\mathfrak{s}}, q^*_{i\mathfrak{s}}, \rho)|_{\mathfrak{s}+\tau} \\ &\quad + \mathcal{H}_{x'_\tau}(\mathfrak{s}, x^*_\mathfrak{s}, x^*_{\tau\mathfrak{s}}, x^{*'}_\mathfrak{s}, x^{*'}_{\tau\mathfrak{s}}, v^*_\mathfrak{s}, v^*_{\tau\mathfrak{s}}, p^*_{i\mathfrak{s}}, q^*_{i\mathfrak{s}}, \rho)|_{\mathfrak{s}+\tau}] \Big] \Big\} \\ &\quad + q^*_{i\mathfrak{s}} d\mathcal{B}_\mathfrak{s},\ \mathfrak{s} \in [0, T], \\ p^*_{iT} &= \varphi^i_0 \mathbb{E}'\Big[k_{ix}(x^*_T, x^{*'}_T) + k_{ix'}(x^*_T, x^{*'}_T) \Big] \\ &\quad + \sum_{j \neq i} \varphi^i_j \mathbb{E}'\Big[k_{jx}(x^*_T, x^{*'}_T) + k_{jx'}(x^*_T, x^{*'}_T) \Big], \\ p^*_{i\mathfrak{s}} &= 0,\ \mathfrak{s} \in (T, T+\tau],\ q^*_{i\mathfrak{s}} = 0,\ \mathfrak{s} \in [T, T+\tau], \end{aligned} \tag{9}$$

and $\varphi^i_0 \geq 0,\ |\varphi^i_0|^2 + \sum_{j \neq i} |\varphi^i_j|^2 = 1,$

$$\begin{aligned} &\sum_{j \neq i} \varphi^i_j (z_j - c^\star_j) \leq 0,\ \forall z \in (-\infty, c^\star_1] \times \cdots \times (-\infty, c^\star_{i-1}] \\ &\times (-\infty, c^\star_{i+1}] \times \cdots (-\infty, c^\star_M]. \end{aligned} \tag{10}$$

From (10), it follows that $\varphi^i_j \geq 0,\ j \in \bar{M}\backslash i$. Let $a = \sum_{i=1}^M (\varphi^i_0 + \sum_{j \neq i} \varphi^i_j)$, then it is easy to see that $a > 0$, and define $\rho_i = \frac{\varphi^i_0 + \sum_{j \neq i} \varphi^i_j}{a}$. Clearly, $0 < \rho_i < 1$ and $\sum_{i=1}^M \rho_i = 1$. Let $p^* = \frac{\sum_{i=1}^M p^*_i}{a}$, $q^* = \frac{\sum_{i=1}^M q^*_i}{a}$, and $\mathcal{H} = \frac{\sum_{i=1}^M \mathcal{H}^i}{a}$. Basic computations demonstrate that (4) and (6) hold. ■

Following this, we will demonstrate that the combination of maximum principle [16] and certain convexity conditions provide sufficient criteria for achieving Pareto efficiency.

*Theorem 2.6 (**Sufficient Conditions**):* Let (x^*, u^*) denote an admissible pair which is given and the assumptions **(H1)-(H3)** hold. Suppose there exist $\rho = (\rho_1, \rho_2, \cdots, \rho_M) \in \mathcal{C}$, and $(p^*, q^*) \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \times \mathfrak{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ represents the solution of (6). Assume $\mathcal{H}(\mathfrak{s}, \cdots, p^*_{\mathfrak{s}}, q^*_{\mathfrak{s}}, \rho)$ and $k := \sum_{i=1}^{M} \rho_i k_i$ are convex with respect to their respective variables respectively, for any $\mathfrak{s} \in [0,T]$. If (4) is satisfied, then u^* is a POC.

Proof: Define $\mathcal{J}_\rho(v) := \sum_{i=1}^{M} \rho_i \mathcal{J}_i(v)$ and for each $v \in \mathcal{U}$,

$$\begin{aligned}
&\mathcal{J}_\rho(v) - \mathcal{J}_\rho(u^*)\\
&= \sum_{i=1}^{M} \rho_i \mathtt{E}\Bigg\{ \int_0^T \Bigg(\mathtt{E}'\Big[l_i(\mathfrak{s}, x_{\mathfrak{s}}, x'_{\mathfrak{s}}, v_{\mathfrak{s}}, v_{\tau\mathfrak{s}}) \Big] - \mathtt{E}'\Big[l_i(\mathfrak{s}, x^*_{\mathfrak{s}}, x^{*'}_{\mathfrak{s}}, u^*_{\mathfrak{s}}, u^*_{\tau\mathfrak{s}}) \Big] \Bigg)\\
&\quad \times d\mathfrak{s} + \mathtt{E}'\Big[k_i(x_T, x'_T) \Big] - \mathtt{E}'\Big[k_i(x^*_T, x^{*'}_T) \Big] \Bigg\}\\
&= \mathtt{E}\Bigg\{ \int_0^T \Bigg(\mathtt{E}'\Big[\mathcal{H}(\mathfrak{s}, x, x_\tau, x', x'_\tau, p^*, q^*, \rho)\\
&\quad - \mathcal{H}(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, p^*, q^*, \rho) \Big]\\
&\quad - \mathtt{E}'\Big[\Big\langle p^*, f(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau)\\
&\quad - f(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau) \Big\rangle \Big]\\
&\quad - \mathtt{E}'\Big[\Big\langle q^*, \sigma(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau)\\
&\quad - \sigma(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau) \Big\rangle \Big] \Bigg) d\mathfrak{s} + \mathtt{E}'\Big[k(x_T, x'_T)\\
&\quad - k(x^*_T, x^{*'}_T) \Big] \Bigg\} = I_1 - I_2 - I_3 + I_4, \text{where} \qquad (11)
\end{aligned}$$

$$\begin{aligned}
I_1 &= \mathtt{E} \int_0^T \mathtt{E}'\Big[\mathcal{H}(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau, p^*, q^*, \rho)\\
&\quad - \mathcal{H}(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau, p^*, q^*, \rho) \Big] d\mathfrak{s}\\
I_2 &= \mathtt{E} \int_0^T \mathtt{E}'\Big[\Big\langle p^*, f(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau)\\
&\quad - f(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau) \Big\rangle \Big] d\mathfrak{s}\\
I_3 &= \mathtt{E} \int_0^T \mathtt{E}'\Big[\Big\langle q^*, \sigma(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau)\\
&\quad - \sigma(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau) \Big\rangle \Big] d\mathfrak{s}\\
I_4 &= \mathtt{E}\Big[\mathtt{E}'\Big[k(x_T, x'_T) - k(x^*_T, x^{*'}_T) \Big] \Big]
\end{aligned}$$

Since $\mathcal{H}(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau, p^*, q^*, \rho)$ is convex with respect to $(x, x_\tau, x', x'_\tau, v, v_\tau)$, we have

$$\begin{aligned}
I_1 \geq \mathtt{E} \int_0^T \mathtt{E}' \Big[&\Big\langle \mathcal{H}_x(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau, p^*, q^*, \rho), x - x^* \Big\rangle\\
&+ \Big\langle \mathcal{H}_{x_\tau}(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau, p^*, q^*, \rho), x_\tau - x^*_\tau \Big\rangle\\
&+ \Big\langle \mathcal{H}'_x(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau, p^*, q^*, \rho), x' - x^{*'} \Big\rangle\\
&+ \Big\langle \mathcal{H}_{x'_\tau}(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau, p^*, q^*, \rho), x'_\tau - x^{*'}_\tau \Big\rangle\\
&+ \Big\langle \mathcal{H}_v(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau, p^*, q^*, \rho), v - u^* \Big\rangle\\
&+ \Big\langle \mathcal{H}_{v_\tau}(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau, p^*, q^*, \rho), v_\tau - u^*_\tau \Big\rangle \Big] d\mathfrak{s}. \qquad (12)
\end{aligned}$$

Likewise, as $k(x, x')$ is convex with respect to x, x',

$$\begin{aligned}
I_4 &\geq \mathtt{E}\Big[\mathtt{E}'\Big[\Big\langle k_x(x^*_T, x^{*'}_T), x_T - x^*_T \Big\rangle\\
&\qquad + \Big\langle k'_x(x^*_T, x^{*'}_T), x'_T - x^{*'}_T \Big\rangle \Big] \Big]\\
&\geq \mathtt{E}\Big[\mathtt{E}'\Big[\Big\langle k_x(x^*_T, x^{*'}_T) + k'_x(x^*_T, x^{*'}_T), x_T - x^*_T \Big\rangle \Big] \Big]. \qquad (13)
\end{aligned}$$

Using Itô's formula on $\mathtt{E}\Big[\langle p^*, x - x^* \rangle \Big]$, we derive

$$\begin{aligned}
I_4 \geq \mathtt{E} \int_0^T \mathtt{E}' \Big\{ &\Big\langle p^*, f(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau)\\
&- f(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau) \Big\rangle\\
&+ \Big\langle q^*, \sigma(\mathfrak{s}, x, x_\tau, x', x'_\tau, v, v_\tau)\\
&- \sigma(\mathfrak{s}, x^*, x^*_\tau, x^{*'}, x^{*'}_\tau, u^*, u^*_\tau) \Big\rangle\\
&- \Big\langle x - x^*, \mathcal{H}_x(\mathfrak{s}, x^*, x^*_\tau, x', x'_\tau, u^*, u^*_\tau, p^*, q^*, \rho)\\
&+ \mathtt{E}^{\mathcal{F}_\mathfrak{s}}\Big[\mathcal{H}_{x_\tau}(\mathfrak{s}, x^*, x^*_\tau, x', x'_\tau, u^*, u^*_\tau, p^*, q^*, \rho)|_{\mathfrak{s}+\tau} \Big]\\
&+ \mathcal{H}_{x'}(\mathfrak{s}, x^*, x^*_\tau, x', x'_\tau, u^*, u^*_\tau, p^*, q^*, \rho)\\
&+ \mathtt{E}^{\mathcal{F}_\mathfrak{s}}\Big[\mathcal{H}_{x'_\tau}(\mathfrak{s}, x^*, x^*_\tau, x', x'_\tau, u^*, u^*_\tau, p^*, q^*, \rho)|_{\mathfrak{s}+\tau} \Big] \Big\rangle \Big\} d\mathfrak{s}. \qquad (14)
\end{aligned}$$

By combining (13) and (14), we derive

$$\begin{aligned}
&\mathcal{J}_\rho(v) - \mathcal{J}_\rho(u^*)\\
&\geq \mathtt{E} \int_0^T \mathtt{E}' \Big[\Big\langle \mathcal{H}_v(\mathfrak{s}, x^*, x^*_\tau, x', x'_\tau, u^*, u^*_\tau, p^*, q^*, \rho), v - u^* \Big\rangle\\
&+ \Big\langle \mathcal{H}_{v_\tau}(\mathfrak{s}, x^*, x^*_\tau, x', x'_\tau, u^*, u^*_\tau, p^*, q^*, \rho), v_\tau - u^*_\tau \Big\rangle \Big] d\mathfrak{s}. \qquad (15)
\end{aligned}$$

The optimal condition (4) implies $\mathcal{J}_\rho(v) \geq \mathcal{J}_\rho(u^*)$, indicating that $\sum_{i=1}^{M} \rho_i \mathcal{J}_i(v)$ achieves its minimum at u^*. Lemma 2.2 asserts that, u^* is a Pareto optimal control. ■

III. Linear Quadratic Optimal Control Problem

This section explores an LQ-OCP as an application of the theoretical results from the previous section.

Consider the following linear MF-SDDE:

$$\begin{aligned} dx_\mathfrak{s} = &\Big[A_\mathfrak{s}x_\mathfrak{s} + \widehat{A}_\mathfrak{s}\mathtt{E}[x_\mathfrak{s}] + \widetilde{A}_\mathfrak{s}x_{\tau\mathfrak{s}} + \bar{A}_\mathfrak{s}\mathtt{E}[x_{\tau\mathfrak{s}}] + \sum_{i=1}^{M} B_{i\mathfrak{s}}u_{i\mathfrak{s}} \\ &+ \widetilde{B}_{i\mathfrak{s}}u_{i\tau\mathfrak{s}}\Big]d\mathfrak{s} + \Big[C_\mathfrak{s}x_\mathfrak{s} + \widehat{C}_\mathfrak{s}\mathtt{E}[x_\mathfrak{s}] + \widetilde{C}_\mathfrak{s}x_{\tau\mathfrak{s}} + \bar{C}_\mathfrak{s}\mathtt{E}[x_{\tau\mathfrak{s}}] \\ &+ \sum_{i=1}^{M} D_{i\mathfrak{s}}u_{i\mathfrak{s}}\Big]d\mathcal{B}_\mathfrak{s},\ \mathfrak{s} \in [0,T], \\ x_\mathfrak{s} = \xi_\mathfrak{s},\ u_{i\mathfrak{s}} = \eta_{i\mathfrak{s}},\ i \in \mathcal{C},\ \mathfrak{s} \in [-\tau, 0], \end{aligned} \tag{16}$$

where $x_\mathfrak{s}, x_{\tau\mathfrak{s}} \in \mathbb{R}$ represent the state vector and the delayed state vector with delay τ, respectively. $u_{i\mathfrak{s}},\ i \in \bar{M}$ denotes the control strategy of the i^{th} player. Initial paths $\xi_\mathfrak{s},\ \eta_{i\mathfrak{s}}$ are given for the state and control processes, respectively. The admissible control set $\mathcal{U}$ is defined as $\mathcal{U} = \mathfrak{L}^2_\mathcal{F}(0,T;\mathbb{R}) \times \mathfrak{L}^2_\mathcal{F}(0,T;\mathbb{R}) \cdots \times \mathfrak{L}^2_\mathcal{F}(0,T;\mathbb{R})$.

The following assumption **(H4)** on the system's coefficients ensures the existence and uniqueness of solutions to the linear MF-SDDE (16).

Assumption (H4): The functions $A, \widehat{A}, \widetilde{A}, \bar{A}, C, \widehat{C}, \widetilde{C}, \bar{C}, B_i, \widetilde{B}_i, D_i \in \mathfrak{L}^\infty_\mathcal{F}(0,T;\mathbb{R}),\ i \in \bar{M}$. The initial paths $\xi_\mathfrak{s} \in C(-\tau, 0;\mathbb{R})$ and $\eta_{i\mathfrak{s}} \in C(-\tau, 0;\mathbb{R}),\ i \in \bar{M}$.

The cost functional for each player $i,\ i \in \bar{M}$, is specified as follows:

$$\begin{aligned} &\mathcal{J}_i(u_1, u_2, \cdots, u_M, \xi_\mathfrak{s}) \\ &= \frac{1}{2}\mathtt{E}\Big\{\int_0^T \Big[x_\mathfrak{s}^\top \mathcal{V}_{i\mathfrak{s}}x_\mathfrak{s} + (\mathtt{E}[x_\mathfrak{s}])^\top \widehat{\mathcal{V}}_{i\mathfrak{s}}\mathtt{E}[x_\mathfrak{s}] + \sum_{j=1}^{M}\Big(u_{j\mathfrak{s}}^\top \mathcal{W}_{ij\mathfrak{s}}u_{j\mathfrak{s}} \\ &+ u_{j\tau\mathfrak{s}}^\top \widetilde{\mathcal{W}}_{ij\mathfrak{s}}u_{j\tau\mathfrak{s}}\Big)\Big]d\mathfrak{s} + x_T^\top \mathcal{V}_{iT}x_T + (\mathtt{E}[x_T])^\top \widehat{\mathcal{V}}_{iT}\mathtt{E}[x_T]\Big\}. \end{aligned} \tag{17}$$

Assumption (H5): The functions $\mathcal{V}_i, \widehat{\mathcal{V}}_i, \mathcal{W}_{ij}, \widetilde{\mathcal{W}}_{ij} \in \mathfrak{L}^\infty(0,T;\mathbb{S}),\ i,j \in \bar{M}$. $\mathcal{V}_{iT}, \widehat{\mathcal{V}}_{iT} \in \mathbb{S},\ i \in \bar{M}$ and $\exists\ \kappa_1, \kappa_2 > 0$, such that $\mathcal{V}_{i\mathfrak{s}},\ \widehat{\mathcal{V}}_{i\mathfrak{s}},\ \mathcal{V}_{iT},\ \widehat{\mathcal{V}}_{iT} \geq 0,\ \mathcal{W}_{ij} \geq \kappa_1 I, \widetilde{\mathcal{W}}_{ij} \geq \kappa_2 I,\ i,j \in \bar{M}$, for a.e. $\mathfrak{s} \in [0,T]$.

The assumption **(H5)** on the cost functional's coefficients ensures the cost remains finite and the optimization problem is well-posed, with stable solutions and well-defined integrals.

To simplify, we establish the following notations: $B := (B_1, B_2, \cdots, B_M),\ \widetilde{B} := (\widetilde{B}_1, \widetilde{B}_2, \cdots, \widetilde{B}_M)$, $D := (D_1, D_2, \cdots, D_M),\ v = (u_1^\top, \cdots, u_M^\top)^\top,\ v_\tau = (u_{1\tau}^\top, \cdots, u_{M\tau}^\top)^\top,\ \mathcal{W}_i := diag\{\mathcal{W}_{i1}, \cdots, \mathcal{W}_{iM}\},\ \widetilde{\mathcal{W}}_i := diag\{\widetilde{\mathcal{W}}_{i1}, \cdots, \widetilde{\mathcal{W}}_{iM}\},\ i \in \bar{M}$.

With the aforementioned notations, we can reformulate the proposed linear system (16) and the cost functional (17) as:

$$\begin{aligned} dx_\mathfrak{s} = &\Big[A_\mathfrak{s}x_\mathfrak{s} + \widehat{A}_\mathfrak{s}\mathtt{E}[x_\mathfrak{s}] + \widetilde{A}x_{\tau\mathfrak{s}} + \bar{A}\mathtt{E}[x_{\tau\mathfrak{s}}] + B_\mathfrak{s}v_\mathfrak{s} + \widetilde{B}_\mathfrak{s}v_{\tau\mathfrak{s}}\Big]d\mathfrak{s} \\ &+ \Big[C_\mathfrak{s}x_\mathfrak{s} + \widehat{C}_\mathfrak{s}\mathtt{E}[x_\mathfrak{s}] + \widetilde{C}_\mathfrak{s}x_{\tau\mathfrak{s}} + \bar{C}_\mathfrak{s}\mathtt{E}[x_{\tau\mathfrak{s}}] \\ &+ D_\mathfrak{s}v_\mathfrak{s}\Big]d\mathcal{B}_\mathfrak{s},\ \mathfrak{s} \in [0,T], \\ x_\mathfrak{s} = \xi_\mathfrak{s},\ v_\mathfrak{s} = (\eta_{1\mathfrak{s}}^\top \cdots, \eta_{M\mathfrak{s}}^\top)^\top,\ \mathfrak{s} \in [-\tau, 0], \end{aligned} \tag{18}$$

and for $i \in \bar{M}$,

$$\begin{aligned} \mathcal{J}_i(v, \xi_\mathfrak{s}) = \frac{1}{2}\mathtt{E}\Big\{\int_0^T \Big[&x_\mathfrak{s}^\top \mathcal{V}_{i\mathfrak{s}}x_\mathfrak{s} + (\mathtt{E}[x_\mathfrak{s}])^\top \widehat{\mathcal{V}}_{i\mathfrak{s}}\mathtt{E}[x_\mathfrak{s}] + v_\mathfrak{s}^\top \mathcal{W}_{i\mathfrak{s}}v_\mathfrak{s} \\ &+ v_{\tau\mathfrak{s}}^\top \widetilde{\mathcal{W}}_{i\mathfrak{s}}v_{\tau\mathfrak{s}}\Big]d\mathfrak{s} + x_T^\top \mathcal{V}_{iT}x_T + (\mathtt{E}[x_T])^\top \widehat{\mathcal{V}}_{iT}\mathtt{E}[x_T]\Big\}. \end{aligned} \tag{19}$$

Lemma 3.1 ([11]): Assuming **(H4)-(H5)** are satisfied, and for any $\beta \in (0,1),\ v_1, v_2 \in \mathcal{U}$ and $\rho \in \mathcal{C}$, the following condition holds: $\mathcal{J}_i(\beta v_1 + (1-\beta)v_2) \leq \beta\mathcal{J}_i(v_1) + (1-\beta)\mathcal{J}_i(v_2), i \in \bar{M}$.

Remark 3.2: Lemma 2.2 indicates that solving a weighted sum minimization problem is sufficient to determine all Pareto optimal strategies for the LQ problem.

Hence, in order to derive the Pareto optimal strategy of the LQ problem, we write the expression for cost functional with weights is as follows:

$$\begin{aligned} \mathcal{J}_\rho(v) = \sum_{i=1}^{M}\rho_i\mathcal{J}_i(v) = \frac{1}{2}\mathtt{E}\Big[\int_0^T \Big(&x_\mathfrak{s}^\top \mathcal{V}_{\rho\mathfrak{s}}x_\mathfrak{s} + (\mathtt{E}[x_\mathfrak{s}])^\top \widehat{\mathcal{V}}_{\rho\mathfrak{s}}\mathtt{E}[x_\mathfrak{s}] \\ &+ v_\mathfrak{s}^\top \mathcal{W}_{\rho\mathfrak{s}}v_\mathfrak{s} + v_{\tau\mathfrak{s}}^\top \widetilde{\mathcal{W}}_{\rho\mathfrak{s}}v_{\tau\mathfrak{s}}\Big)d\mathfrak{s} + x_T^\top \mathcal{V}_{\rho T}x_T + (\mathtt{E}[x_T])^\top \widehat{\mathcal{V}}_{\rho T} \\ &\times \mathtt{E}[x_T]\Big], \end{aligned}$$

where $\mathcal{V}_\rho := \sum_{i=1}^{M}\rho_i\mathcal{V}_i,\ \widehat{\mathcal{V}}_\rho := \sum_{i=1}^{M}\rho_i\widehat{\mathcal{V}}_i,\ \mathcal{W}_\rho := \sum_{i=1}^{M}\rho_i\mathcal{W}_i,\ \widetilde{\mathcal{W}}_\rho := \sum_{i=1}^{M}\rho_i\widetilde{\mathcal{W}}_i,\ \mathcal{V}_{T\rho} := \sum_{i=1}^{M}\rho_i\mathcal{V}_{iT},\ \widehat{\mathcal{V}}_{T\rho} := \sum_{i=1}^{M}\rho_i\widehat{\mathcal{V}}_{iT}$.

The primary goal of the optimization problem with weights is to identify $v^* \in \mathcal{U}$ which minimizes the above cost functional $\mathcal{J}_\rho(v)$. That is, $\mathcal{J}_\rho(v^*) = \min_{v\in\mathcal{U}}\mathcal{J}_\rho(v)$. A strategy is considered as Pareto if one such v^* exists and the associated solution is denoted by $(\mathcal{J}_1(v^*), \cdots, \mathcal{J}_M(v^*))$. This minimization problem is known as problem **(MF-DP$_{LQ}$)**.

Based on the necessary and sufficient conditions outlined in earlier section, we establish the following theorem for the LQ Pareto game.

Theorem 3.3: Given that **(H4)-(H5)** are satisfied, and v^* is a POC for problem **(MF-DP$_{LQ}$)**, if and only if v^* fulfills

the optimality condition given below:

$$v_{\mathfrak{s}}^{*}\mathcal{W}_{\rho\mathfrak{s}}+v_{\tau\mathfrak{s}}^{*}\widetilde{\mathcal{W}}_{\rho\mathfrak{s}}+B_{\mathfrak{s}}^{\top}p_{\mathfrak{s}}^{*}+\widetilde{B}_{\mathfrak{s}}^{\top}p_{\mathfrak{s}}^{*}+D_{\mathfrak{s}}^{\top}q_{\mathfrak{s}}^{*}=0, \quad (20)$$

where $(x_{\mathfrak{s}}^{*},p_{\mathfrak{s}}^{*},q_{\mathfrak{s}}^{*})$ satisfies

$$\begin{aligned}
dx_{\mathfrak{s}}^{*}=&\Big[A_{\mathfrak{s}}x_{\mathfrak{s}}^{*}+\widehat{A}_{\mathfrak{s}}\mathbb{E}[x_{\mathfrak{s}}^{*}]+\widetilde{A}x_{\tau\mathfrak{s}}^{*}+\bar{A}\mathbb{E}[x_{\tau\mathfrak{s}}^{*}]+B_{\mathfrak{s}}v_{\mathfrak{s}}^{*}+\widetilde{B}_{\mathfrak{s}}v_{\tau\mathfrak{s}}^{*}\Big]d\mathfrak{s}\\
&+\Big[C_{\mathfrak{s}}x_{\mathfrak{s}}^{*}+\widehat{C}_{\mathfrak{s}}\mathbb{E}[x_{\mathfrak{s}}^{*}]+\widetilde{C}_{\mathfrak{s}}x_{\tau\mathfrak{s}}^{*}+\bar{C}_{\mathfrak{s}}\mathbb{E}[x_{\tau\mathfrak{s}}^{*}]+D_{\mathfrak{s}}v_{\mathfrak{s}}^{*}\Big]\\
&\times d\mathcal{B}_{\mathfrak{s}},\ \mathfrak{s}\in[0,T],\\
dp_{\mathfrak{s}}^{*}=&-\Big[\mathcal{V}_{\rho\mathfrak{s}}x_{\mathfrak{s}}^{*}+\widehat{\mathcal{V}}_{\rho\mathfrak{s}}\mathbb{E}[x_{\mathfrak{s}}^{*}]+A_{\mathfrak{s}}^{\top}p_{\mathfrak{s}}^{*}+\widehat{A}_{\mathfrak{s}}^{\top}\mathbb{E}[p_{\mathfrak{s}}^{*}]+C_{\mathfrak{s}}^{\top}q_{\mathfrak{s}}^{*}\\
&+\mathbb{E}^{\mathcal{F}_{\mathfrak{s}}}\Big[(\widetilde{A}_{\mathfrak{s}}^{\top}p_{\mathfrak{s}}^{*}+\bar{A}_{\mathfrak{s}}^{\top}\mathbb{E}[p_{\mathfrak{s}}^{*}]+\widehat{C}_{\mathfrak{s}}^{\top}q_{\mathfrak{s}}^{*})|_{\mathfrak{s}+\tau}\Big]\Big]d\mathfrak{s}+q_{\mathfrak{s}}^{*}d\mathcal{B}_{\mathfrak{s}},\\
x_{\mathfrak{s}}^{*}=&\xi_{\mathfrak{s}},\ v_{\mathfrak{s}}^{*}=(\eta_{1\mathfrak{s}}^{\top}\cdots,\eta_{M\mathfrak{s}}^{\top})^{\top},\ \mathfrak{s}\in[-\tau,0],\\
p_{T}^{*}=&\mathcal{V}_{\rho T}x_{T}^{*}+\widehat{\mathcal{V}}_{\rho T}\mathbb{E}[x_{T}^{*}],\\
p_{\mathfrak{s}}^{*}=&0,\ \mathfrak{s}\in(T,T+\tau],q_{\mathfrak{s}}^{*}=0,\ \mathfrak{s}\in[T,T+\tau]. \quad (21)
\end{aligned}$$

Remark 3.4: The control derived in (20) is an open-loop optimal control. Typically, in LQ problems, we aim to derive the optimal control in a closed-loop form. The feedback representation of optimal control is widely recognized as valuable for practical applications. In the classical case, the optimal feedback control is expressed through the Riccati equation. However, for mean-field stochastic systems with delay, determining the feedback control becomes challenging due to the dependence on the system's history and the mean-field interactions. Therefore, we focus only on deriving the open-loop optimal control in this context. For more details on feedback control in delay systems, readers are encouraged to refer to [23].

IV. CONCLUSIONS

In this study, we developed both sufficient and necessary conditions for achieving POC in a CDG driven by SDDE of mean-field type over a finite time. The Pareto optimality of the proposed system is achieved by converting the Pareto OCP to a set of constrained mean-field type stochastic OCPs with delay and then applying the maximum principle to show that these conditions are met. Moreover, sufficient conditions are demonstrated under certain convexity assumptions. At last, an LQ problem that incorporated the theoretical results was addressed. The authors aim to investigate the Pareto game under a non-convex control domain, with a future focus on deriving feedback control for mean-field stochastic delay systems.

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