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Anti-Windup Compensator Design of Continuous Time-Delayed Systems with Actuator Saturation using Guaranteed Cost Control Approach

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***Abstract*— This paper focuses on designing the anti-windup (AW) compensator for continuous time-delayed systems with actuator saturation (AS) using the guaranteed cost control (GCC) approach. The study specifically considers linear time-delayed systems with AS and takes into account both the AW design and stability analysis of the system. To compute the AW compensator gain, a linear matrix inequality (LMI)-based study is used. Some suitable examples are used as a demonstration of the approach presented.**

***Keywords*: Anti-windup, delayed systems, guaranteed cost control**

I. Introduction

Actuator saturation (AS) nonlinearity is frequent observed in engineering systems, affecting the performance of the system. These nonlinearities arise due to physical constraints or safety requirements imposed on the actuators. The AS can negatively impact the dynamic behavior of the system, leading to reduced performance. This creates a noticeable effect on the normal functioning of the system, potentially causing disruptions or deviations from desired behavior. The design and analysis of systems with AS have been subjects of considerable theoretical and practical importance, leading to significant research efforts in recent decades [1, 2, 3, 5, 7, 9, 10, 14].

Time delays typically occur in a variety of real-world control systems, including chemical, power, and communication, etc. When delays are present, systems can become unstable and performance can be deteriorated. Delay partitioning, free weighting matrices, and Wirtinger inequality approaches are a few of the efficient techniques that have been developed to deal with delayed systems [3, 4, 5, 7, 11, 15, 16].

One of the major problems in controller design is wind-up. In a PID controller, a large change in set point arises due to which integral terms sum up to a significant rise, consequently overshooting and continuing to increase as this sum up error is unbounded. This situation refers to the integral windup. The main concern of the windup is excess overshooting. This problem occurs in both linear and non-linear systems [11].

To compensate for this windup effect, the AW technique is used. A nominal controller (often linear) that has AW compensation attached results in an AW compensator. AW compensation has the benefit of not affecting the loop until saturation does not take place. As a result, when the control signals are within the saturation limits, the control action of the AW controller is the same as that of the nominal controller. The design can be divided into two parts, with the linear controller taking up the first and the AW modification taking up the second.

In [10], a linear controller is given by ignoring control input nonlinearities, and then an AW bumpless transfer compensator is included. In [9], for systems with AS, an observer-based AW compensator is formed. The Iterative optimization using linear matrix inequality (LMI) constraints is used to define and get the AW compensator gain for linear systems in [2].

Guaranteed cost control (GCC) is the robust control strategy used to deal with the systems with uncertainties and nonlinearities. It focused on designing controller which provides the upper bound on the system performance index along with the stability. In [12], the GCC for cyber-physical assembly system is introduced. The AW controller design for many systems is widely discussed in the literature [1–11, 13, 16]. However, to the best of authors' knowledge, the AW design using GCC for delayed systems is not mentioned previously. The delay-dependent approach to study the AW controller using GCC is an important problem in practice.

Motivated by above, this paper presents an AW compensator design based on GCC approach for delayed systems subjected to AS. The Lyapunov function method is employed to form LMIs that assure the stability of closed-loop systems in the presence of AS. Furthermore, an optimization problem is formulated to reduce the upper bound of the cost function associated with the system. This optimization aims to improve the system performance by minimizing the cost associated with AS and achieving more desirable control outcomes. The AW compensator gain for the time-delayed systems with uncertainties is also presented.

The paper is arranged as follows. In Section II, the preliminary concepts and problem formulation necessary for the design of the compensator are presented. A GCC-based AW compensator for systems with time delay is introduced by utilizing an LMI-based criterion in Section III. To demonstrate the efficacy of the designed controller, some

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relevant examples are provided in Section IV. Finally in Section V, the conclusion of the paper is given.

II. Problem Formulation

The linear time delayed system is considered as

$$\dot{\boldsymbol{x}}_s(t) = \boldsymbol{A}_s\,\boldsymbol{x}_s(t) + \boldsymbol{A}_{ds}\,\boldsymbol{x}_s(t-\tau) + \boldsymbol{B}_s\,sat\big(\boldsymbol{u}(t)\big), \tag{1a}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}_s\,\boldsymbol{x}_s(t), \tag{1b}$$

where state vector is $\boldsymbol{x}_s(t)\in\Re^p$, input is $\boldsymbol{u}(t)\in\Re^q$ and output is $\boldsymbol{y}(t)\in\Re^r$. Matrices $\boldsymbol{A}_s\in\Re^{p\times p}$, $\boldsymbol{A}_{ds}\in\Re^{p\times p}$, $\boldsymbol{B}_s\in\Re^{p\times q}$, $\boldsymbol{C}_s\in\Re^{r\times p}$ are known constant matrices.

The dynamic output stabilizing compensator without considering AS is expressed as follows

$$\dot{\boldsymbol{x}}_\varepsilon(t) = \boldsymbol{A}_\varepsilon\,\boldsymbol{x}_\varepsilon(t) + \boldsymbol{B}_\varepsilon\,\boldsymbol{u}_\varepsilon(t), \tag{2a}$$

$$\boldsymbol{v}_\varepsilon(t) = \boldsymbol{C}_\varepsilon\,\boldsymbol{x}_\varepsilon(t) + \boldsymbol{D}_\varepsilon\,\boldsymbol{u}_\varepsilon(t), \tag{2b}$$

where $\boldsymbol{x}_\varepsilon(t)\in\Re^{p_\varepsilon}$ is the controller state, $\boldsymbol{u}_\varepsilon(t)=\boldsymbol{y}(t)$ and $\boldsymbol{v}_\varepsilon(t)=\boldsymbol{u}(t)$. Matrices $\boldsymbol{A}_\varepsilon\in\Re^{p_\varepsilon\times p_\varepsilon}$, $\boldsymbol{B}_\varepsilon\in\Re^{p_\varepsilon\times r}$, $\boldsymbol{C}_\varepsilon\in\Re^{q\times p_\varepsilon}$, $\boldsymbol{D}_\varepsilon\in\Re^{q\times r}$ are known matrices. The system (1) with AS is

$$-u_{0_{(k)}} \le v_{(k)} \le u_{o_{(k)}}, \tag{3}$$

where $u_{0_{(k)}}>0,\ k=1,2,...,q$ indicates the control limits. The saturated control signal is represented as follows

$$\boldsymbol{u}(t) = sat\big(\boldsymbol{v}_\varepsilon(t)\big) = sat\big(\boldsymbol{C}_\varepsilon\,\boldsymbol{x}_\varepsilon(t) + \boldsymbol{D}_\varepsilon\,\boldsymbol{u}_\varepsilon(t)\big). \tag{4}$$

The AW controller incorporates a rectification term that is used to eliminate the adverse consequences of windup caused by AS which is defined as $\boldsymbol{F}_\varepsilon\big(sat(\boldsymbol{v}_\varepsilon(t)) - \boldsymbol{v}_\varepsilon(t)\big)$. Consequently, the closed-loop system is

$$\dot{\boldsymbol{\zeta}}(t) = \boldsymbol{A}\boldsymbol{\zeta}(t) + \boldsymbol{A}_d\,\boldsymbol{\zeta}(t-\tau) - (\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)\,\phi\big(\boldsymbol{L}\boldsymbol{\zeta}(t)\big), \tag{5a}$$

where

$$\boldsymbol{\zeta}(t) = \begin{bmatrix}\boldsymbol{x}_s(t)\\ \boldsymbol{x}_\varepsilon(t)\end{bmatrix},\ \boldsymbol{A} = \begin{bmatrix}\boldsymbol{A}_s + \boldsymbol{B}_s\,\boldsymbol{D}_\varepsilon\boldsymbol{C}_s & \boldsymbol{B}_s\boldsymbol{C}_\varepsilon\\ \boldsymbol{B}_\varepsilon\boldsymbol{C}_s & \boldsymbol{A}_\varepsilon\end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix}\boldsymbol{B}_s\\ \boldsymbol{0}\end{bmatrix}, \tag{5b}$$

$$\boldsymbol{A}_d = \begin{bmatrix}\boldsymbol{A}_{ds} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix},\ \boldsymbol{G} = \begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}_{nc}\end{bmatrix},\ \boldsymbol{L} = \begin{bmatrix}\boldsymbol{D}_\varepsilon\boldsymbol{C}_s & \boldsymbol{C}_\varepsilon\end{bmatrix}. \tag{5c}$$

The function $\phi\big(\boldsymbol{L}\boldsymbol{\zeta}(t)\big)$ refers to the decentralized dead-zone non-linearity and is represented as

$$\phi\big(\boldsymbol{L}\boldsymbol{\zeta}(t)\big) = \boldsymbol{v}_\varepsilon(t) - sat\big(\boldsymbol{v}_\varepsilon(t)\big) = \boldsymbol{L}\boldsymbol{\zeta}(t) - sat\big(\boldsymbol{L}\boldsymbol{\zeta}(t)\big). \tag{6}$$

The cost function associated with (6) is defined as follows,

$$J = \int_0^\infty \boldsymbol{\zeta}^T(t)\,\boldsymbol{Q}_1\,\boldsymbol{\zeta}(t) + \Big(sat\big(\boldsymbol{v}_\varepsilon(t)\big)\Big)^T\,\boldsymbol{Q}_2\,sat\big(\boldsymbol{v}_\varepsilon(t)\big)\,dt, \tag{7}$$

where $\boldsymbol{Q}_1>\boldsymbol{0}$ and $\boldsymbol{Q}_2\ge\boldsymbol{0}$. Let $\boldsymbol{0}\le\boldsymbol{P}\in\Re^{(p+p_\varepsilon)\times(p+p_\varepsilon)}$ and $\rho>0$ be a scalar. The ellipsoid $\Omega(\boldsymbol{P},\rho)$ is defined as

$$\Omega(\boldsymbol{P},\rho) = \big\{\boldsymbol{\zeta}(t)\in\Re^{p+p_\varepsilon} : \boldsymbol{\zeta}^T(t)\,\boldsymbol{P}\,\boldsymbol{\zeta}(t)\le\rho\big\}. \tag{8}$$

The polyhedron is described as:

$$\ell(\boldsymbol{L}_s,\boldsymbol{U}_s) = \big\{\boldsymbol{\zeta}(t)\in\Re^{p+p_\varepsilon} : \big|(\boldsymbol{L}_s-\boldsymbol{U}_s)\,\boldsymbol{\zeta}(t)\big|\le 1,\ s=1,2,...,q\big\} \tag{9}$$

where $\boldsymbol{L}_s$ and $\boldsymbol{U}_s$ are s-th row of the matrix $\boldsymbol{L}$ and $\boldsymbol{U}$, respectively.

Lemma 1 [8]: Consider the function $\phi\big(\boldsymbol{L}\boldsymbol{\zeta}(t)\big)$ defined in (9). Then, the following

$$\phi^T\big(\boldsymbol{L}\boldsymbol{\zeta}(t)\big)\,\boldsymbol{M}\Big[\phi\big(\boldsymbol{L}\boldsymbol{\zeta}(t)\big) - \boldsymbol{U}\boldsymbol{\zeta}(t)\Big] \le \boldsymbol{0}, \tag{10}$$

holds for any diagonal matrix $\boldsymbol{M}>\boldsymbol{0}$.

Lemma 2 [19, 20]: For any scalar $\lambda>0$ and matrices $\boldsymbol{H}_o, \boldsymbol{F}, \boldsymbol{N}_o$ of the appropriate dimensions, then

$$\boldsymbol{X} + \boldsymbol{H}_o\boldsymbol{F}\boldsymbol{N}_o + (\boldsymbol{H}_o\boldsymbol{F}\boldsymbol{N}_o)^T < \boldsymbol{0} \tag{11}$$

holds if and only if

$$\boldsymbol{X} + \lambda\boldsymbol{H}_o\boldsymbol{H}_o^T + \lambda^{-1}\boldsymbol{N}_0^T\boldsymbol{N}_o < \boldsymbol{0}, \tag{12}$$

where $\boldsymbol{F}^T\boldsymbol{F}\le\boldsymbol{I}$.

III. Main Results

In this section, we utilize the concepts of Lyapunov function and LMI theory to derive conditions ensuring the existence of AW controller based on GCC. This control law aims to achieve asymptotic stability for system (5) while providing an upper bound for the associated cost function (7). Additionally, we propose an optimization approach to design optimal AW compensator. The AW results for uncertain delayed systems are also there in this section.

A. AW design for the systems without uncertainty:

Theorem 1: Consider a closed-loop system (5) and the cost function (7), then there exists an output-feedback AW controller (2) based on GCC, if there are scalars $\tau>0$, $\rho>0$, symmetric and positive-definite matrices $\boldsymbol{W}\in\Re^{(p+p_\varepsilon)\times(p+p_\varepsilon)}$, $\boldsymbol{X}\in\Re^{(p+p_\varepsilon)\times(p+p_\varepsilon)}$, $\boldsymbol{H}\in\Re^{(p+p_\varepsilon)\times(p+p_\varepsilon)}$, $\boldsymbol{S}\in\Re^{(p+p_\varepsilon)\times(p+p_\varepsilon)}$, a positive diagonal matrix $\boldsymbol{Q}\in\Re^{q\times q}$ and matrices $\boldsymbol{Y}\in\Re^{q\times(p+p_\varepsilon)}$, $\boldsymbol{K}\in\Re^{q\times(p+p_\varepsilon)}$, $\boldsymbol{Z}\in\Re^{p_\varepsilon\times q}$ such that the LMIs

$$\begin{bmatrix}
\boldsymbol{\Psi} & \boldsymbol{\Lambda} & \sqrt{\tau}\boldsymbol{W}\boldsymbol{A}^T & \sqrt{\tau}\boldsymbol{W}\boldsymbol{A}_d^T & \boldsymbol{W}^T & \boldsymbol{Y}^T & \boldsymbol{0}\\
\boldsymbol{\Lambda}^T & -2\boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{Q} & \sqrt{\tau}(\boldsymbol{B}\boldsymbol{Q}+\boldsymbol{G}\boldsymbol{Z})^T\\
\sqrt{\tau}\boldsymbol{A}\boldsymbol{W} & \boldsymbol{0} & -\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\
\sqrt{\tau}\boldsymbol{A}_d\boldsymbol{W} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{S} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\
\boldsymbol{W} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Q}_1^{-1} & \boldsymbol{0} & \boldsymbol{0}\\
\boldsymbol{Y} & \boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Q}_2^{-1} & \boldsymbol{0}\\
\boldsymbol{0} & \sqrt{\tau}(\boldsymbol{B}\boldsymbol{Q}+\boldsymbol{G}\boldsymbol{Z}) & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{H}
\end{bmatrix} \le \boldsymbol{0}, \tag{13a}$$

$$\Psi = \boldsymbol{W}(\boldsymbol{A}+\boldsymbol{A}_d)^T + (\boldsymbol{A}+\boldsymbol{A}_d)\boldsymbol{W} + \tau\boldsymbol{A}_d(\boldsymbol{H}+\boldsymbol{X}+\boldsymbol{S})\boldsymbol{A}_d^T, \tag{13b}$$

$$\Lambda = \boldsymbol{B}\boldsymbol{Q} + \boldsymbol{G}\boldsymbol{Z} - \boldsymbol{K}^T, \tag{13c}$$

$$\begin{bmatrix} \boldsymbol{W} & \boldsymbol{Y}_s^T - \boldsymbol{K}_s^T \\ \boldsymbol{Y}_s - \boldsymbol{K}_s & \rho^2 \end{bmatrix} \geq \boldsymbol{0}, \ s = 1, 2, ..., q \tag{14}$$

have feasible solution. In this case, the AW compensator gain with guaranteed cost is $\boldsymbol{F}_\varepsilon = \boldsymbol{Z}\boldsymbol{Q}^{-1}$ and the cost function (7) meets

$$J < trace\,\boldsymbol{P} + \tau trace\left(\boldsymbol{N}_1^T \boldsymbol{A}^T \boldsymbol{X}^{-1} \boldsymbol{A}\boldsymbol{N}_1\right) + \tau trace\left(\boldsymbol{N}_1^T \boldsymbol{A}_d^T \boldsymbol{S}^{-1} \boldsymbol{A}_d \boldsymbol{N}_1\right)$$
$$+\tau trace\left(\boldsymbol{N}_2^T (\boldsymbol{B}\boldsymbol{Q} + \boldsymbol{G}\boldsymbol{Z})^T \boldsymbol{H}^{-1} (\boldsymbol{B}\boldsymbol{Q} + \boldsymbol{G}\boldsymbol{Z}) \boldsymbol{N}_2\right) = J^*. \tag{15}$$

Proof: Consider the Lyapunov function given as

$$V(\zeta(t)) = \zeta(t)^T \boldsymbol{P}\zeta(t) + \dot{\Gamma}(\zeta(t)), \tag{16a}$$

$$\Gamma(\zeta(t)) = \int_{-\tau}^{0}\int_{t+\gamma}^{t} \left(\zeta^T(\theta)\boldsymbol{A}^T X^{-1}\boldsymbol{A}\zeta(\theta)\right) d\theta d\gamma$$
$$+\int_{-\tau}^{0}\int_{t-\tau+\gamma}^{t} \left(\zeta^T(\theta)\boldsymbol{A}_d^T \boldsymbol{S}^{-1}\boldsymbol{A}_d\zeta(\theta)\right) d\theta d\gamma$$
$$+\int_{-\tau}^{0}\int_{t+\gamma}^{\tau} \left(\phi^T(\boldsymbol{L}\zeta(\theta))(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)^T \boldsymbol{H}^{-1}\right.$$
$$\left.\times(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)\phi(\boldsymbol{L}\zeta(\theta))\right) d\theta d\gamma \tag{16b}$$

Now taking

$$\mu(\zeta(t)) = -\int_{-\tau}^{0} \zeta^T(t)\boldsymbol{P}\boldsymbol{A}_d\boldsymbol{A}\zeta(t+\gamma)d\gamma$$
$$-\int_{-\tau}^{0} \zeta^T(t+\gamma)\boldsymbol{A}^T\boldsymbol{A}_d^T\boldsymbol{P}\zeta(t)d\gamma, \tag{17a}$$

$$\nu(\zeta(t)) = -\int_{-\tau}^{0} \zeta^T(t)\boldsymbol{P}\boldsymbol{A}_d\boldsymbol{A}_d\zeta(t+\gamma-\tau)d\gamma$$
$$-\int_{-\tau}^{0} \zeta^T(t+\gamma-\tau)^T\boldsymbol{A}_d^T\boldsymbol{A}_d^T\boldsymbol{P}\zeta(t)d\gamma, \tag{17b}$$

$$\chi(\zeta(t)) = \int_{-\tau}^{0} \zeta^T(t)\boldsymbol{P}\boldsymbol{A}_d(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)\phi(\boldsymbol{L}\zeta(t+\gamma))d\gamma$$
$$+\int_{-\tau}^{0} \left(\phi(\boldsymbol{L}\zeta(t+\gamma))^T(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)^T\boldsymbol{A}_d^T\boldsymbol{P}\zeta(t)\right)d\gamma. \tag{17c}$$

From $2u^Tv \leq u^T\boldsymbol{\Sigma}u + v^T\boldsymbol{\Sigma}^{-1}v$, we can state that

$$\mu(\zeta(t)) \leq \tau\zeta^T(t)\boldsymbol{P}\boldsymbol{A}_d\boldsymbol{X}\boldsymbol{A}_d^T\boldsymbol{P} + \int_{-\tau}^{0} \zeta^T(t+\gamma)\boldsymbol{A}^T\boldsymbol{X}^{-1}\boldsymbol{A}\zeta(t+\gamma)d\gamma, \tag{18a}$$

$$\nu(\zeta(t)) \leq \tau\zeta^T(t)\boldsymbol{P}\boldsymbol{A}_d\boldsymbol{S}\boldsymbol{A}_d^T + \int_{-\tau}^{0} \zeta^T(t+\gamma-\tau)\boldsymbol{A}_d^T\boldsymbol{S}^{-1}\boldsymbol{A}_d\zeta(t+\gamma-\tau)d\gamma, \tag{18b}$$

$$\chi(\zeta(t)) \leq \tau\zeta^T(t)\boldsymbol{P}\boldsymbol{A}_d\boldsymbol{H}\boldsymbol{A}_d^T\boldsymbol{P}\zeta(t)$$
$$+\int_{-\tau}^{0} \phi^T(\boldsymbol{L}\zeta(t+\gamma))(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)^T\boldsymbol{H}^{-1}(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)\zeta(t+\gamma)\phi(\boldsymbol{L}\zeta(t+\gamma))d\gamma. \tag{18c}$$

Now, by taking the derivative of (16) along the trajectory of (5) and by using Lemma 1, we get

$$\dot{V}(\zeta(t)) \leq \zeta^T(t)\left((\boldsymbol{A}+\boldsymbol{A}_d)^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}+\boldsymbol{A}_d)\right)\zeta(t) - \zeta^T(t)\boldsymbol{P}$$
$$\times(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)\phi(\boldsymbol{L}\zeta(t)) - \phi^T(\boldsymbol{L}\zeta(t))(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)^T\boldsymbol{P}\zeta(t)$$
$$+\mu(\zeta(t)) + \nu(\zeta(t)) + \chi(\zeta(t)) + \dot{\Gamma}\zeta(t)$$
$$-2\phi^T(\boldsymbol{L}\zeta(t))\boldsymbol{M}\phi(\boldsymbol{L}\zeta(t)) + 2\phi^T(\boldsymbol{L}\zeta(t))\boldsymbol{M}\boldsymbol{U}\zeta(t). \tag{19}$$

From (18) and (19), we get

$$\dot{V}(\zeta(t)) + \zeta^T(t)\boldsymbol{Q}_1\zeta(t) + (sat(\boldsymbol{v}_c))^T\boldsymbol{Q}_2 sat(\boldsymbol{v}_c) \leq$$
$$\zeta^T(t)\left((\boldsymbol{A}+\boldsymbol{A}_d)^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}+\boldsymbol{A}_d)\right)\zeta(t) - \zeta^T(t)\boldsymbol{P}(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)$$
$$\phi(\boldsymbol{L}\zeta(t)) - \phi^T(\boldsymbol{L}\zeta(t))(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)^T\boldsymbol{P}\zeta(t) + \tau\zeta^T(t)\boldsymbol{P}\boldsymbol{A}_d$$
$$\boldsymbol{X}\boldsymbol{A}_d^T\boldsymbol{P}\zeta(t) + \tau\zeta^T(t)\boldsymbol{A}^T\boldsymbol{X}^{-1}\boldsymbol{A}\zeta(t) + \tau\zeta^T(t)\boldsymbol{P}\boldsymbol{A}_d\boldsymbol{S}$$
$$\boldsymbol{A}_d^T\boldsymbol{P}\zeta(t) + \tau\zeta^T(t)\boldsymbol{A}_d^T\boldsymbol{S}^{-1}\boldsymbol{A}_d\zeta(t) + \tau\zeta^T(t)\boldsymbol{P}\boldsymbol{A}_d\boldsymbol{H}\boldsymbol{A}_d^T$$
$$\boldsymbol{P}\zeta(t) + \tau\phi^T(\boldsymbol{L}\zeta(t))(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)^T\boldsymbol{H}^{-1}(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)\phi(\boldsymbol{L}\zeta(t))$$
$$-2\phi^T(\boldsymbol{L}\zeta(t))\boldsymbol{M}\phi(\boldsymbol{L}\zeta(t)) + 2\phi^T(\boldsymbol{L}\zeta(t))\boldsymbol{M}\boldsymbol{U}\zeta(t)$$
$$+\zeta^T(t)\boldsymbol{Q}_1\zeta(t) + \{\boldsymbol{L}\zeta(t) - \phi(\boldsymbol{L}\zeta(t))\}^T$$
$$\boldsymbol{Q}_2\{\boldsymbol{L}\zeta(t) - \phi(\boldsymbol{L}\zeta(t))\}. \tag{20}$$

Equation (20) can be

$$\dot{V}(\zeta(t)) + \zeta^T(t)\boldsymbol{Q}_1\zeta(t) + (sat(\boldsymbol{v}_c))^T\boldsymbol{Q}_2 sat(\boldsymbol{v}_c) \leq$$
$$\begin{bmatrix}\zeta^T(t) & -\phi^T(\boldsymbol{L}\zeta(t))\end{bmatrix}\boldsymbol{E}\begin{bmatrix}\zeta(t) \\ -\phi(\boldsymbol{L}\zeta(t))\end{bmatrix}, \tag{21a}$$

where

$$\boldsymbol{E} = \begin{bmatrix} \boldsymbol{\eta} & \boldsymbol{P}(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon) - \boldsymbol{U}^T\boldsymbol{M}^T + \boldsymbol{L}^T\boldsymbol{Q}_2 \\ (\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)^T\boldsymbol{P} - \boldsymbol{M}\boldsymbol{U} + \boldsymbol{Q}_2\boldsymbol{L} & \tau(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon)^T\boldsymbol{H}^{-1}(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_\varepsilon) - 2\boldsymbol{M} + \boldsymbol{Q}_2 \end{bmatrix}, \tag{21b}$$

$$\boldsymbol{\eta} = (\boldsymbol{A}+\boldsymbol{A}_d)^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}+\boldsymbol{A}_d) + \tau\boldsymbol{P}\boldsymbol{A}_d\boldsymbol{X}\boldsymbol{A}_d^T\boldsymbol{P} + \tau\boldsymbol{A}^T\boldsymbol{X}^{-1}\boldsymbol{A} +$$
$$\tau\boldsymbol{P}\boldsymbol{A}_d\boldsymbol{S}\boldsymbol{A}_d^T\boldsymbol{P} + \tau\boldsymbol{A}_d^T\boldsymbol{S}^{-1}\boldsymbol{A}_d + \tau\boldsymbol{P}\boldsymbol{A}_d\boldsymbol{H}\boldsymbol{A}_d^T\boldsymbol{P} + \boldsymbol{Q}_1 + \boldsymbol{L}^T\boldsymbol{Q}_2\boldsymbol{L} \tag{21c}$$

Observe that, $\dot{V}(\zeta(t)) < 0$ if $\boldsymbol{E} < \boldsymbol{0}$.

On pre and post multiplying equation $\boldsymbol{E} < \mathbf{0}$ by $diag\left[\boldsymbol{P}^{-1}, \boldsymbol{M}^{-1}\right]$, and considering $\boldsymbol{P}^{-1}=\boldsymbol{W}$, $\boldsymbol{M}^{-1}=\boldsymbol{Q}$, $\boldsymbol{F}_{\varepsilon}=\boldsymbol{Z}\boldsymbol{Q}^{-1}$, $\boldsymbol{Y}=\boldsymbol{L}\boldsymbol{P}^{-1}$, $\boldsymbol{K}=\boldsymbol{U}\boldsymbol{P}^{-1}$ we get (13) using Schur's complement.

The LMI (14) represents that the ellipsoid (9) is a subset of polyhedral set (10) with $\boldsymbol{Y}_s=\boldsymbol{L}_s\boldsymbol{P}^{-1}$ and $\boldsymbol{K}_s=\boldsymbol{U}_s\boldsymbol{P}^{-1}$.

From $\boldsymbol{E} < \mathbf{0}$, it follows that

$$\boldsymbol{\zeta}^T(t)\boldsymbol{Q}_1\boldsymbol{\zeta}(t)+\left(sat(\boldsymbol{v}_c)\right)^T\boldsymbol{Q}_2\,sat(\boldsymbol{v}_c)\le -\dot{V}\left(\boldsymbol{\zeta}(t)\right). \tag{22}$$

On integrating both sides of (23) from 0 to ∞, we get

$$\int_0^{\infty}\boldsymbol{\zeta}(t)^T\boldsymbol{Q}_1\boldsymbol{\zeta}(t)+\left(sat(\boldsymbol{v}_c)\right)^T\boldsymbol{Q}_2\,sat(\boldsymbol{v}_c)\,dt\le V\left(\boldsymbol{\zeta}(0)\right). \tag{23}$$

From (7) and (23), one obtains

$$\begin{aligned}J &\le V\left(\boldsymbol{\zeta}(0)\right)=\boldsymbol{\zeta}^T(0)\boldsymbol{P}\boldsymbol{\zeta}(0)+\boldsymbol{Q}\left(\boldsymbol{\zeta}(0)\right)\\ &=\boldsymbol{\zeta}^T(0)\boldsymbol{P}\boldsymbol{\zeta}(0)+\int_{-\tau}^{0}\int_{\gamma}^{0}\left(\boldsymbol{\zeta}^T(\theta)\boldsymbol{A}^T\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{\zeta}(\theta)\right)d\theta d\gamma\\ &+\int_{-\tau}^{0}\int_{-\tau+\gamma}^{0}\left(\boldsymbol{\zeta}^T(\theta)\boldsymbol{A}_d{}^T\boldsymbol{S}^{-1}\boldsymbol{A}_d\boldsymbol{\zeta}(\theta)\right)d\theta d\gamma+\int_{-\tau}^{0}\int_{\gamma}^{0}\left(\phi^T\left(\boldsymbol{L}\boldsymbol{\zeta}(\theta)\right)\right.\\ &\left.\times\left(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_{\varepsilon}\right)^T\boldsymbol{H}^{-1}\left(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_{\varepsilon}\right)\phi\left(\boldsymbol{L}\boldsymbol{\zeta}(\theta)\right)\right)d\theta d\gamma\end{aligned} \tag{24}$$

From (24), we can see that the upper bound is dependent on initial conditions. To remove this dependence, let

$$\boldsymbol{\zeta}^T(0)\boldsymbol{P}\boldsymbol{\zeta}(0)=trace(\boldsymbol{P}), \tag{25a}$$

$$\int_{-\tau}^{0}\boldsymbol{\zeta}^T(\theta)\boldsymbol{\zeta}(\theta)d\theta=\boldsymbol{N}_1\boldsymbol{N}_1{}^T, \tag{25b}$$

$$\int_{-\tau}^{0}\phi^T\left(\boldsymbol{L}\boldsymbol{\zeta}(\theta)\right)\phi\left(\boldsymbol{L}\boldsymbol{\zeta}(\theta)\right)d\theta=\boldsymbol{N}_2\boldsymbol{N}_2{}^T. \tag{25c}$$

Then,

$$\begin{aligned}&\int_{-\tau}^{0}\int_{\gamma}^{0}\left(\boldsymbol{\zeta}^T(\theta)\boldsymbol{A}^T\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{\zeta}(\theta)\right)d\theta d\gamma<\tau\int_{-\tau}^{0}\left(\boldsymbol{\zeta}^T(\theta)\boldsymbol{A}^T\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{\zeta}(\theta)\right)d\theta d\gamma\\ &=\tau\,trace\left(\boldsymbol{N}_1{}^T\boldsymbol{A}^T\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{N}_1\right)\end{aligned} \tag{26a}$$

$$\begin{aligned}&\int_{-\tau}^{0}\int_{-\tau+\gamma}^{0}\left(\boldsymbol{\zeta}^T(\theta)\boldsymbol{A}_d{}^T\boldsymbol{S}^{-1}\boldsymbol{A}_d\boldsymbol{\zeta}(\theta)\right)d\theta d\gamma<\tau\int_{-\tau}^{0}\left(\boldsymbol{\zeta}^T(\theta)\boldsymbol{A}_d{}^T\boldsymbol{S}^{-1}\boldsymbol{A}_d\boldsymbol{\zeta}(\theta)\right)d\theta d\gamma\\ &=\tau\,trace\left(\boldsymbol{N}_1{}^T\boldsymbol{A}_d{}^T\boldsymbol{S}^{-1}\boldsymbol{A}_d\boldsymbol{N}_1\right)\end{aligned} \tag{26b}$$

$$\begin{aligned}&\int_{-\tau}^{0}\int_{\gamma}^{0}\left(\phi^T\left(\boldsymbol{L}\boldsymbol{\zeta}(\theta)\right)\left(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_{\varepsilon}\right)^T\boldsymbol{H}^{-1}\left(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_{\varepsilon}\right)\phi\left(\boldsymbol{L}\boldsymbol{\zeta}(\theta)\right)\right)d\theta\,d\gamma\\ &<\tau\int_{-\tau}^{0}\phi^T\left(\boldsymbol{L}\boldsymbol{\zeta}(\theta)\right)\boldsymbol{M}^{-1}\left(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_{\varepsilon}\right)^T\boldsymbol{H}^{-1}\left(\boldsymbol{B}+\boldsymbol{G}\boldsymbol{F}_{\varepsilon}\right)\boldsymbol{M}^{-1}\phi\left(\boldsymbol{L}\boldsymbol{\zeta}(\theta)\right)d\theta\\ &=\tau\,trace\left(\boldsymbol{N}_2{}^T\left(\boldsymbol{B}\boldsymbol{Q}+\boldsymbol{G}\boldsymbol{Z}\right)^T\boldsymbol{H}^{-1}\left(\boldsymbol{B}\boldsymbol{Q}+\boldsymbol{G}\boldsymbol{Z}\right)\boldsymbol{N}_2\right).\end{aligned} \tag{26c}$$

Using (26a)-(26c), equation (24) leads to (15).

Theorem 1 provides the feasible solution for the dynamic output-feedback AW controller with guaranteed cost. For any sufficient solution to LMIs (12) and (13), an AW compensator gain and upper bound can be obtained. The optimal AW controller that reduces the upper bound of the cost function is more advantageous in practice.

Theorem 2: Consider the system (5) and the cost function (7), there exists an optimal AW controller with guaranteed cost if the following optimization problem

minimize

$$trace\left(\boldsymbol{N}\right)+\tau trace\left(\boldsymbol{M}_1\right)+\tau trace\left(\boldsymbol{M}_2\right)+\tau trace\left(\boldsymbol{M}_3\right)$$

subjected to.

(i). (13),

(ii). (14),

(iii). $\begin{bmatrix}-\boldsymbol{N} & \boldsymbol{I}\\ \boldsymbol{I} & \boldsymbol{W}\end{bmatrix}<\mathbf{0}$, (27a)

(iv). $\begin{bmatrix}-\boldsymbol{M}_1 & \boldsymbol{N}_1\boldsymbol{A}^T\\ \boldsymbol{A}\boldsymbol{N}_1 & -\boldsymbol{X}\end{bmatrix}<\mathbf{0}$, (27b)

(v). $\begin{bmatrix}-\boldsymbol{M}_2 & \boldsymbol{N}_1{}^T\boldsymbol{A}_d{}^T\\ \boldsymbol{A}_d\boldsymbol{N}_1 & -\boldsymbol{S}\end{bmatrix}<\mathbf{0}$, (27c)

(vi). $\begin{bmatrix}-\boldsymbol{M}_3 & \boldsymbol{N}_2{}^T\left(\boldsymbol{B}\boldsymbol{Q}+\boldsymbol{G}\boldsymbol{Z}\right)^T\\ \left(\boldsymbol{B}\boldsymbol{Q}+\boldsymbol{G}\boldsymbol{Z}\right)\boldsymbol{N}_2 & -\boldsymbol{H}\end{bmatrix}<\mathbf{0}$ (27d)

gives a feasible solution for positive-definite symmetric matrix $\boldsymbol{W}\in\Re^{(p+p_{\varepsilon})\times(p+p_{\varepsilon})}$, $\boldsymbol{X}\in\Re^{(p+p_{\varepsilon})\times(p+p_{\varepsilon})}$, $\boldsymbol{H}\in\Re^{(p+p_{\varepsilon})\times(p+p_{\varepsilon})}$, $\boldsymbol{S}\in\Re^{(p+p_{\varepsilon})\times(p+p_{\varepsilon})}$, a positive-definite diagonal matrix $\boldsymbol{Q}\in\Re^{q\times q}$, matrices $\boldsymbol{Y}\in\Re^{q\times(p+p_{\varepsilon})}$, $\boldsymbol{Z}\in\Re^{p_{\varepsilon}\times q}$ and positive-definite matrices $\boldsymbol{N}$, $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{M}_3$, of suitable dimensions. In this case, the optimal AW compensator gain matrix is $\boldsymbol{F}_{\varepsilon}=\boldsymbol{Z}\boldsymbol{Q}^{-1}$ and the upper limit of the cost function satisfies

$$\begin{aligned}J &< trace\,\boldsymbol{P}+\tau\,trace\left(\boldsymbol{N}_1{}^T\boldsymbol{A}^T\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{N}_1\right)+\tau\,trace\\ &\left(\boldsymbol{N}_1{}^T\boldsymbol{A}_d{}^T\boldsymbol{S}^{-1}\boldsymbol{A}_d\boldsymbol{N}_1\right)+\tau\,trace\left(\boldsymbol{N}_2{}^T\left(\boldsymbol{B}\boldsymbol{Q}+\boldsymbol{G}\boldsymbol{Z}\right)^T\boldsymbol{H}^{-1}\left(\boldsymbol{B}\boldsymbol{Q}+\boldsymbol{G}\boldsymbol{Z}\right)\boldsymbol{N}_2\right)=J_{opt}^{*}\end{aligned} \tag{28}$$

Proof: Let $trace\left(\boldsymbol{P}\right)<trace\left(\boldsymbol{N}\right)\Rightarrow\boldsymbol{P}-\boldsymbol{N}<\mathbf{0}$ which gives $\begin{bmatrix}-\boldsymbol{N} & \boldsymbol{I}\\ \boldsymbol{I} & \boldsymbol{P}^{-1}\end{bmatrix}\le\mathbf{0}$ by Schur's complement and by considering $\boldsymbol{P}^{-1}=\boldsymbol{W}$, (27a) is obtained. Similarly,

$$\tau\,trace\left(\boldsymbol{N}_1{}^T\boldsymbol{A}^T\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{N}_1\right)<\tau\,trace\left(\boldsymbol{M}_1\right). \tag{29}$$

results in (27b),

$$\tau\,trace\left(\boldsymbol{N}_1{}^T\boldsymbol{A}_d{}^T\boldsymbol{S}^{-1}\boldsymbol{A}_d\boldsymbol{N}_1\right)<\tau\,trace\left(\boldsymbol{M}_2\right). \tag{30}$$

results in (27c), and

$$\tau trace\left(\boldsymbol{N}_2^{\,T}\left(\boldsymbol{BQ}+\boldsymbol{GZ}\right)^T\boldsymbol{H}^{-1}\left(\boldsymbol{BQ}+\boldsymbol{GZ}\right)\boldsymbol{N}_2\right)<\tau trace\left(\boldsymbol{M}_3\right) \tag{31}$$

results in (27d).

B. AW design for the systems with uncertainty:

Consider a continuous-time uncertain system

$$\dot{\boldsymbol{x}}_s(t)=(\boldsymbol{A}+\Delta\boldsymbol{A})\boldsymbol{x}_s(t)+(\boldsymbol{A}_d+\Delta\boldsymbol{A}_d)x_s(t-\tau)+\boldsymbol{B}sat\boldsymbol{u}(t) \tag{32a}$$

$$\boldsymbol{y}(t)=\boldsymbol{C}\boldsymbol{x}_s(t) \tag{32b}$$

where $\Delta\boldsymbol{A}=\boldsymbol{HFN}_A$ and $\Delta\boldsymbol{A}_d=\boldsymbol{HFN}_{A_d}$ are uncertain matrices. Matrices $\boldsymbol{H}$, $\boldsymbol{N}_A$, and $\boldsymbol{N}_{A_d}$ are known matrices of suitable dimensions and an unknown matrix $\boldsymbol{F}$ satisfies $\boldsymbol{F}^T\boldsymbol{F}\le\boldsymbol{I}$. The dynamic output feedback controller is given in (2). From (32) and (2), the closed-loop system leads to

$$\dot{x}_c\left(t\right)=\boldsymbol{A}_c\boldsymbol{x}_c(t)+\boldsymbol{B}_c\boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{F}_c(sat(\boldsymbol{v}_c(t))-\boldsymbol{v}_c(t)), \tag{33}$$

$$\boldsymbol{A}_o=\begin{bmatrix}(\boldsymbol{A}+\Delta\boldsymbol{A})+\boldsymbol{B}\boldsymbol{D}_c\boldsymbol{C}+(\boldsymbol{A}_d+\Delta\boldsymbol{A}_d) & \boldsymbol{B}\boldsymbol{C}_c\\ \boldsymbol{B}_c & \boldsymbol{A}_c\end{bmatrix},\ \boldsymbol{B}_o=\begin{bmatrix}\boldsymbol{B}\\ \boldsymbol{0}\end{bmatrix}$$

$$\boldsymbol{k}=\begin{bmatrix}\boldsymbol{C}\boldsymbol{D}_c\\ \boldsymbol{C}_c\end{bmatrix}^T,\ \boldsymbol{R}=\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}_n\end{bmatrix},\ \boldsymbol{A}_{d_o}=\begin{bmatrix}(\boldsymbol{A}_d+\Delta\boldsymbol{A}_d) & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}.$$

Theorem 3: Let the closed-loop system (33) and the cost function (5), then a dynamic output feedback controller based on GCC exists for $\tau>0$, $\lambda>0$, positive definite and symmetric matrices $\boldsymbol{P}\in R^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{X}\in R^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{S}\in R^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{H}\in R^{(n+n_c)\times(n+n_c)}$, positive diagonal matrix $\boldsymbol{T}\in R^{q\times q}$ and matrices $\boldsymbol{\Lambda}\in R^{q\times(n+n_c)}$, $\boldsymbol{\gamma}\in R^{q\times(n+n_c)}$, $\boldsymbol{Z}\in R^{n_c\times q}$ such that the LMIs

$$\begin{bmatrix}
\boldsymbol{W}(\boldsymbol{A}_o+\boldsymbol{A}_{do})^T+(\boldsymbol{A}_o+\boldsymbol{A}_{do})\boldsymbol{W} & (\boldsymbol{B}_o\boldsymbol{T}+\boldsymbol{RZ})-\boldsymbol{\gamma}^T & \sqrt{\tau}\boldsymbol{A}_{do} & \sqrt{\tau}\boldsymbol{W}\boldsymbol{A}_o & \sqrt{\tau}\boldsymbol{A}_{do} & \sqrt{\tau}\boldsymbol{W}\boldsymbol{A}_{do}\\
(\boldsymbol{B}_o\boldsymbol{T}+\boldsymbol{RZ})^T-\boldsymbol{\gamma} & -2\boldsymbol{T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\
\sqrt{\tau}\boldsymbol{A}_{do}^T & \boldsymbol{0} & -\boldsymbol{X}^1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\
\sqrt{\tau}\boldsymbol{A}_o^T\boldsymbol{W} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0}\\
\sqrt{\tau}\boldsymbol{A}_{do}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{S}^{-1} & \boldsymbol{0}\\
\sqrt{\tau}\boldsymbol{A}_{do}^T\boldsymbol{W} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{S}\\
\sqrt{\tau}\boldsymbol{A}_{do}^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\
\boldsymbol{0} & \sqrt{\tau}(\boldsymbol{B}_o\boldsymbol{T}+\boldsymbol{RZ}) & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\
\boldsymbol{\Lambda} & \boldsymbol{T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\
\boldsymbol{W} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\
\boldsymbol{H}_o^T\boldsymbol{W} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\
\sqrt{\tau}(\boldsymbol{N}_{Ao}+\boldsymbol{N}_{Ado})\boldsymbol{W} & \boldsymbol{0} & \sqrt{\tau}\boldsymbol{N}_{A_{do}} & \sqrt{\tau}\boldsymbol{N}_{Ao} & \sqrt{\tau}\boldsymbol{N}_{A_{do}} & \sqrt{\tau}\boldsymbol{N}_{A_{do}}
\end{bmatrix}$$

$$\begin{bmatrix}
\sqrt{\tau}\boldsymbol{A}_{do} & \boldsymbol{0} & \boldsymbol{\Lambda} & \boldsymbol{W} & \boldsymbol{W}\boldsymbol{H}_o^T & \sqrt{\tau}\boldsymbol{W}(\boldsymbol{N}_{Ao}+\boldsymbol{N}_{Ado})^T\\
\boldsymbol{0} & \sqrt{\tau}(\boldsymbol{B}_o\boldsymbol{T}+\boldsymbol{RZ})^T & \boldsymbol{T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \sqrt{\tau}\boldsymbol{N}_{A_{do}}^T\\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \sqrt{\tau}\boldsymbol{N}_{A_o}^T\\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \sqrt{\tau}\boldsymbol{N}_{A_{do}}^T\\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \sqrt{\tau}\boldsymbol{N}_{Ado}^T\\
-\boldsymbol{H}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \sqrt{\tau}\boldsymbol{N}_{Ado}^T\\
\boldsymbol{0} & -\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\
\boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Q}_2^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Q}_1^{-1} & \boldsymbol{0} & \boldsymbol{0}\\
\boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda^{-1} & \boldsymbol{0}\\
\sqrt{\tau}\boldsymbol{N}_{Ado} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\lambda
\end{bmatrix}<\boldsymbol{0}, \tag{34}$$

$$\begin{bmatrix}\boldsymbol{W} & \Lambda_i^T-\gamma_i^T\\ \Lambda_i-\gamma_i & \rho^2\end{bmatrix}\ge\boldsymbol{0},\ i=1,2,.....p \tag{35}$$

have feasible solution. In this situation, the AW compensator gain with guaranteed cost is $\boldsymbol{E}_c=\boldsymbol{ZT}^{-1}$ and the upper bound is achieved as

$$J<trace\,\boldsymbol{P}+\tau trace\left(\boldsymbol{N}_1^{\,T}\boldsymbol{A}^T\boldsymbol{X}^{-1}\boldsymbol{A}\boldsymbol{N}_1\right)+\tau trace\left(\boldsymbol{N}_1^{\,T}\boldsymbol{A}_d^{\,T}\boldsymbol{S}^{-1}\boldsymbol{A}_d\boldsymbol{N}_1\right)$$
$$+\tau trace\left(\boldsymbol{N}_2^{\,T}\left(\boldsymbol{BQ}+\boldsymbol{GZ}\right)^T\boldsymbol{H}^{-1}\left(\boldsymbol{BQ}+\boldsymbol{GZ}\right)\boldsymbol{N}_2\right)=J^* \tag{36}$$

Proof: Taking The Lyapunov function (16), using Lemmas 1, 2 and following all of those steps given in Theorem 1, Theorem 3 can be obtained.

Remark 1: Theorem 1-3 only provides sufficient conditions for the AW compensator gain. More study is required to reduce the difference between sufficiency and necessity using better characterization of delays and uncertainties.

IV. ILLUSTRATIVE EXAMPLES

Example 1: The system described by (1) along with the (2) and the cost function (8) with

$$\boldsymbol{A}_s=\begin{bmatrix}-1 & 1.5\\ 0.3 & -2\end{bmatrix},\ \boldsymbol{A}_{sp}=\begin{bmatrix}0.6 & 0.4\\ 0 & -0.5\end{bmatrix},\ \boldsymbol{B}_s=\begin{bmatrix}10 & 0\\ 0 & 1\end{bmatrix},$$
$$\boldsymbol{C}_s=\begin{bmatrix}5 & 1\\ 0 & 1\end{bmatrix},\ \boldsymbol{A}_\varepsilon=\begin{bmatrix}-20.2042 & 2.5216\\ 2.1415 & -4.4821\end{bmatrix},$$

$$\boldsymbol{B}_\varepsilon=diag\left[1.9516,-0.0649\right],\ \boldsymbol{C}_\varepsilon=\begin{bmatrix}-0.9165 & 0.1091\\ 0 & 1\end{bmatrix},$$

$$\boldsymbol{D}_\varepsilon=\begin{bmatrix}-3.3333 & 1\\ -2.16 & 1.1\end{bmatrix},\ \boldsymbol{Q}_1=diag\left[1,1,1,1\right],\ \boldsymbol{Q}_2=diag\left[1,1\right],$$

$\boldsymbol{N}_1=diag\left[1,1,1,1\right]$, $\boldsymbol{N}_2=diag\left[1,1\right]$, $\tau=1$, $\rho=2$.

By using MATLAB and YALMIP, Theorem 1 is found to be feasible and the AW compensator gain is $\boldsymbol{F}_\varepsilon=\begin{bmatrix}-0.0544 & 0.0016\\ 0.0479 & -0.0178\end{bmatrix}$ and $J^*=6.3051\times10^3$. The optimization problem given in Theorem 2 provides the optimal solution where the optimal AW compensator gain is $\boldsymbol{F}_\varepsilon=\begin{bmatrix}21.6061 & -1.6971\\ -1.5973 & -1.2401\end{bmatrix}$ and $J_{opt}^*=46.4457$. The state trajectories of the closed loop system are represented by Fig. 1. It can be observed from Figure 1 that $\boldsymbol{\zeta}\left(t\right)\to\boldsymbol{0}$ as $t\to\infty$. Control law applied to the system is depicted by Fig. 2.

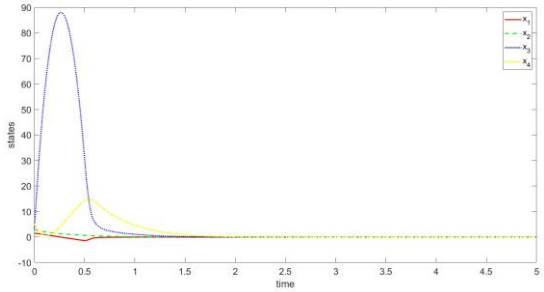


Fig. 1. Plot of state trajectories

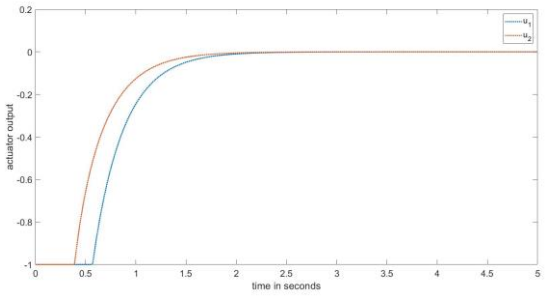


Fig. 2. Control effort

Example 2: Consider (33) with:

$$\boldsymbol{A}=\begin{bmatrix}-0.01 & 0\\ 0 & -0.01\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},\ \boldsymbol{C}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},$$

$$\boldsymbol{A}_c=\begin{bmatrix}0.002 & 0.014\\ 0.002 & -0.0048\end{bmatrix},\ \boldsymbol{B}_c=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},\ \boldsymbol{D}_c=\begin{bmatrix}-3 & 0\\ 0 & 1\end{bmatrix},$$

$$\boldsymbol{C}_c=\begin{bmatrix}0.333 & 0\\ 0 & -0.1\end{bmatrix},\tau=1,\ \rho=1.$$

By using MATLAB and YALMIP, the AW compensator gain is $\boldsymbol{E}_c=\begin{bmatrix}-0.0544 & 0.0016\\ 0.0479 & -0.0178\end{bmatrix}$ and $J^*=20.5040$.

Remark 2: Existing approaches of AW control [2, 4, 5, 7, 9, 13, 15] do not take GCC into consideration. Therefore, these can not be used for GCC based AW design for the above given examples. The presented approach can be more useful for designing AW control for real-world systems.

V. CONCLUSION

The AW controller design for continuous time-delayed systems based on GCC approach is discussed in this paper. The proposed approach is a delay-dependent approach for systems subjected to AS and stability conditions are realized using Lyapunov criteria. The results for AW design for the system in presence of uncertainty is also given. The presented approach is validated using a suitable example. The presented methodology can be further extended to the GCC based AW design for linear systems with time-varying delays.

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