2024 Tenth Indian Control Conference (ICC)
9 - 11 December, 2024
IISER Bhopal, India

State & Parameter Estimation in Lossy Distribution Networks: Algorithmic Perspective with Convergence Certificates

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Abstract— **Insufficient instrumentation and uncertainty in network information hinder the task of voltage estimation in electrical distribution networks. Considering this challenge, in this work, we propose a model-based algorithm that estimates the state (*voltage*) and the parameters (*network impedance*) of a single-phased radial power distribution network. We model this network leveraging the recently introduced *lossy distribution flow equation*. We formulate the estimation problem in the spirit of a constrained optimization framework, which exploits available measurements of network voltage and bus power injection data as ground truth and generates fast voltage and impedance estimates. We theoretically verify the convergence of voltage and impedance estimation errors embracing results from control Lyapunov function and perturbation theory. We simulate our algorithm on the standard IEEE 33-bus distribution system and observe that it provides improvements in both voltage and impedance estimates, in comparison with the traditional power flow models that ignore network losses.**

I. Introduction

In developing countries, insufficient deployment of Phasor Measurement Units (PMUs) substantially impedes the observability of the rural distribution networks [1]. These networks are typically operated in a tree-like structure (i.e. radial topology), while the (residential) loads connected to these networks are single-phased [2]. Due to the rapid integration of (volatile) distributed energy sources and controllable loads, these networks witness unusual fluctuations in the operational conditions (voltage magnitude). Therefore, the key challenge is to generate fast and accurate state estimates (SE) to ensure reliable and optimal operation of the distribution network. The accuracy of the SE algorithms significantly relies on the distribution network models, which include information about the network topology and network impedance (i.e., line resistance and reactance). However, in most cases, these information are unavailable, partially available, or outdated due to system modifications and upgrades [3]. Thus, the challenge of SE comes hand in hand with the secondary challenge to generate reliable parameter estimates (PE) within the physical model of the system. Therefore, in this work, our objective is to devise an optimization-based approach, which can jointly estimate the state and the line parameters of a single-phased radial power distribution network.

SE algorithms are typically formulated as a (nonconvex) *weighted least square* (WLS) optimization problem and solved by leveraging the *Gauss-Newton* (GN) solver, see [4] for an exhaustive survey on this topic. Unlike transmission network, where measurement units are placed at almost all network nodes, the SE task in distribution systems is challenging due to the scarcity of real-time measurements. Likewise, the key problem associated with the GN solver is that it is computationally complex and unscalable [5]. To circumvent these problems, multi-area state estimation technique has been developed in [6], but it only provides approximate solutions to the original WLS problem. Recently, [5] develops a robust multi-area state estimation algorithm considering large errors associated with pseudo-measurements and provides a fast and accurate solution for monitoring system states. However it requires the *exact* network information (topology and impedance values) for solving the SE problems.

As stated earlier, inaccurate network topology and network impedance information diminish the accuracy of the SE algorithms. *Topology identification*, which aims to estimate both the network topology and impedance, is crucial for various distribution system applications [2]. It is worth mentioning that, traditional impedance estimation techniques ignore the effect of continuously changing operational factors [7], and therefore, during different operational conditions, the topology identification task needs to be performed using real-time measurements. Nowadays, smart meters are typically deployed over the distribution network to collect these measurements, which include active, reactive power injection and/or voltage magnitudes [2]. Based on these data, [8] and [9] present a *grid probing* and a *polynomial complexity* algorithm for solving the topology identification problem considering complete and limited network observability. Unlike topology identification, work [10] solves the *impedance estimation* problem, in which one estimates network impedances assuming that the network topology is known. It formulates the problem as a mixed-integer linear programming problem and solves it using the least square techniques. However, it is to be noted that these above works assume a *lossless network model* to estimate the impedances.

Contribution: In this work, we propose a model-based approach for solving the *state and impedance estimation* problem of a single-phased radial power distribution network. To model this network, we adopt the recently introduced (parameterized) *lossy distribution flow equations* [11], [12], which allow us to characterize network losses. We hypothesize that modeling the network losses will yield more accurate state and parameter estimates than the ones obtained using lossless models. We formulate the estimation problem in the framework of a constrained optimization paradigm,

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which exploits bus power injection data and pseudo-voltage measurements as ground truths to obtain the voltage and network impedance estimates. The bus power injection data are obtained via smart-meters installed at the prosumers end, while the pseudo-measurements are forecasted based on the generation and the demand profiles of the network. We leverage results from control Lyapunov function and perturbation theory and certify that the network impedance (resp. voltage) estimation error obtained from this algorithm is bounded (resp. globally exponentially convergent to the origin). We empirically verify the efficacy of this algorithm by implementing it on the standard IEEE 33-bus distribution network and compare its performance with the baseline model, i.e., the *lossless distribution flow* model. We find that the proposed approach elucidates significant improvements in impedance estimation at all loading conditions while reasonably improving the voltage estimation at higher loads.

II. Overview of Lossy Distribution Flow Model

In what follows, we present an abridged version of the (parameterized) *lossy distribution flow model*, see [11], [12] for details. We consider a single-phase radial distribution network, which includes $(n+1)$ number of buses and n number of feeder lines[1]. This class of networks is represented by a tree graph $\mathcal{T} = (\mathcal{N}, \mathcal{E})$, in which $\mathcal{N}$ and $\mathcal{E}$ denote the set of buses, and the set of feeder lines. The topology of this graph is captured by an edge-to-node incidence matrix $\bar{\mathbf{A}} \in \{0, \pm 1\}^{n\times(n+1)}$, which is partitioned into its first and the rest of its columns as $\bar{\mathbf{A}} = \begin{bmatrix} \mathrm{a} & \mathbf{A} \end{bmatrix}$. The elements of $\bar{\mathbf{A}}$ are computed considering the definition from [12]. In this network, the reduced incidence matrix $\mathbf{A} \in \{0, \pm 1\}^{n\times n}$ is square and invertible, and it is obtained by removing the first column of $\bar{\mathbf{A}}$ corresponding to the root node, i.e. $\mathrm{a} \in \{0, 1\}^n$. We consider the substation bus voltage (i.e., root node in $\mathcal{T}$) v_0 is known and fixed, and define $\mathbf{v}_0 = |v_0|^2$. Likewise, the diagonal matrices $(\mathcal{D}_{\mathbf{r}}, \mathcal{D}_{\mathbf{x}})$ accommodate the line resistances and the reactances on their main diagonals. Then, considering the aforesaid network topology and exploiting the lossy distribution flow model [11], [12], we find that the network voltage can be expressed as a linear function of the bus power injections:

$$\mathbf{A}\,(\mathbf{v} - \mathbf{v}_0\mathbf{1}_n) = 2\mathcal{D}_{\mathbf{r}}\mathbf{A}^{-\top}\,\mathbf{p} + 2\mathcal{D}_{\mathbf{x}}\mathbf{A}^{-\top}\,\mathbf{q} \,-\, \Phi\,\mathbf{c}. \quad (1)$$

Here $\Phi = \mathcal{D}^2_{\mathbf{rx}} + 2\mathcal{D}_r\mathbf{A}^{-\top}\mathcal{D}_r + 2\mathcal{D}_x\mathbf{A}^{-\top}\mathcal{D}_x$, in which $\mathcal{D}^2_{\mathbf{rx}} = \mathcal{D}^2_{\mathbf{r}} + \mathcal{D}^2_{\mathbf{x}}$. The vectors $t \mapsto (\mathbf{p}(t), \mathbf{q}(t), \mathbf{v}(t)) \in \mathbb{R}^{3n}$, respectively, collect active, reactive power injection, and the squared voltage magnitude at all buses excluding the substation bus, while $\mathbf{c}(t) \in \mathbb{R}^n$ accommodates the squared current magnitudes of the feeder lines. Note that, in (1), a simplifying assumption is to disregard the ohmic losses on lines by setting $\mathbf{c}(t) = 0$, and, in consequence, (1) transforms into the *lossless distribution flow model* in [2, Eq. (3)]. In contrast, hereinafter, our objective is to model the ohmic losses, albeit in an analytically tractable way, by approximating $\mathbf{c}(t)$. To achieve this, we compute each element of $\mathbf{c}(t)$ as

$$\mathbf{c}_{ij}(t) \,=\, \frac{\mathbf{v}_i + \mathbf{v}_j \,-\, 2v_iv_j\cos\,\delta_{ij}}{r^2_{ij} + x^2_{ij}}. \quad (2)$$

For the kth bus, v_k is the bus voltage and $\mathbf{v}_k = |v_k|^2$. The voltage angle between i and jth bus is $\cos\,\delta_{ij}$, while $(r_{ij},\, x_{ij})$ are the impedance of the ijth feeder. In (2), first we set $\cos\,\delta_{ij} \simeq 1$, considering the *small angle assumption*, and then by revisiting [11], we approximate the mixed term v_iv_j through a convex combination of $\mathbf{v}_i$ and $\mathbf{v}_j$ as

$$v_iv_j \simeq \gamma_{ij}\,\mathbf{v}_i \,+\, (1-\gamma_{ij})\,\mathbf{v}_j, \quad (3)$$

where $\gamma_{ij} \in [0,\,1]$ for all $i, j \in \mathcal{N}$, is known as the *voltage sensitivity function*. Replacing (3) into (2), we obtain $\mathbf{c}(t) = \Psi\,\mathbf{A}\,(\mathbf{v} - v_0\mathbf{1}_n)$, where $\Psi = \left(\mathcal{D}^2_{\mathbf{rx}}\right)^{-1}(\mathbf{I} - 2\mathcal{D}_\gamma)$. Thereafter, substituting $\mathbf{c}(t)$ into (1), we find

$$\mathbf{v} \,=\, \mathfrak{B}\,(\gamma, r, x)\;\mathbf{q} \,+\, \mathfrak{D}\,(\gamma, r, x)\;\mathbf{p} \,+\, \mathbf{v}_0\mathbf{1}_n, \quad (4)$$

where

$$\mathfrak{B}\,(\gamma, r, x) \,=\, 2\mathbf{A}^{-1}\,(\mathbf{I} \,+\, \Phi\Psi)^{-1}\,\mathcal{D}_{\mathbf{x}}\mathbf{A}^{-\top}, \quad (5a)$$

$$\mathfrak{D}\,(\gamma, r, x) \,=\, 2\mathbf{A}^{-1}\,(\mathbf{I} \,+\, \Phi\Psi)^{-1}\,\mathcal{D}_{\mathbf{r}}\mathbf{A}^{-\top}. \quad (5b)$$

Remark 2.1: In (3), if we choose $\gamma_{ij} = 0.5$ for all $i, j \in \mathcal{N}$, then it leads to $(\mathbf{I}_n \,+\, \Phi\Psi) = \mathbf{I}_n$ in (5), where $\mathbf{I}_n$ is the identity matrix of dimension n. Consequently, (4) reduces to

$$\mathbf{v} \,\simeq\, \mathbf{X}\,\mathbf{q} \,+\, \mathbf{R}\,\mathbf{p} \,+\, \mathbf{v}_0\mathbf{1}_n, \quad (6)$$

where $\mathbf{X} = 2\left(\mathbf{A}^{-1}\mathcal{D}_{\mathbf{x}}\mathbf{A}^{-\top}\right)$ and $\mathbf{R} = 2\left(\mathbf{A}^{-1}\mathcal{D}_{\mathbf{r}}\mathbf{A}^{-\top}\right)$. Eqn. (6) describes the *lossless distribution flow* model, and it is considered as a simplified version of (4). Note that, for (6), the task of joint state and parameter estimation is relatively straightforward. For instance, if the ground truth information $(\mathbf{p},\, \mathbf{q},\, \mathbf{v})$ are available of all buses over a given time interval, then $\mathbf{R}$ and $\mathbf{X}$ can easily be estimated due to their simplified diagonal structure, see [2] for details. Furthermore, knowing $(\mathbf{R},\, \mathbf{X})$ and $(\mathbf{p},\, \mathbf{q})$ one can estimate the network voltage employing any "off-the-shelf" optimization techniques.

However, these methods can not be leveraged to solve the same problem for the lossy distribution flow model (4), as its drift matrices $(\mathfrak{B},\, \mathfrak{D})$ exhibit strong coupling between the network resistance and reactance. This motivates us to develop a new framework for solving the state and parameter estimation problem for (4).

III. Proposed Algorithmic Solution

Here we present our proposed approach. It presumes that the subsequent information are available: (i) historical voltage measurements from all buses in the distribution network (since real time measurements may not always be available). (ii) bus power withdrawals/injections data. The task of SE then becomes estimating the network voltages from the aforesaid data. We perform this task in the absence of accurate information about the network impedances, and estimating them as well. The proposed approach is depicted

[1]Note that, if a radial distribution network has $(n+1)$ number of buses, then it explicitly includes n number of feeder lines, due to its tree like structure.

in Figure 1 and formulated in Algorithm 1. Here, over a pre-defined time-period of interest, the objective is to minimize the difference between the estimated and the observed voltage magnitudes of an underlying distribution network, and this deviation is measured by the standard Euclidian norm (7a). The first equality constraint (7b) describes the steady-state map (4) in the presence of estimated network impedances. Note that, in (7a) and (7b), we require $(\mathbf{p}, \mathbf{q}, \mathbf{v})$ of all buses excluding the substation bus. The injection data $(\mathbf{p}, \mathbf{q})$ are obtained either in real-time, from the prosumers end, or can be forecasted with reasonable accuracy. The bus voltage data $\mathbf{v}$ are obtained through direct measurements wherever sensors are available or through pseudo measurements when sensors are not available. The proposed algorithm includes the unknown line impedance values as parameters. The line impedance values are unknown because, in practice, the impedance can deviate from the design values due to aging, faults, vegetation, temperature, etc. Nevertheless, the design values are used to constrain the trend in the deviations, see (7f). In (7b) through (7f), the proposed algorithm estimates the values of network impedances $(\hat{\mathbf{r}}, \hat{\mathbf{x}})$, such that the error between voltage values $\hat{\mathbf{v}}(t)$ determined via the the lossy distribution flow model (with bus power injections – $(\mathbf{p}, \mathbf{q})$, as the inputs) and the observed values $\mathbf{v}(t)$ are minimized. $\hat{\mathbf{v}}(t)$ is considered to be the voltage estimates corrected for measurement noises, if any. From these voltage and the line impedance estimates, other state variables such as line flows, branch currents and voltage angles can be deduced using phasor algebra.

Algorithm 1 State & Parameter estimation algorithm

1: Squared substation bus voltage $\mathbf{v}_0$ ▷ *Input data*
2: Total ground truth samples $\mathcal{S}$ and sample size $|\mathcal{S}|$
3: Resistance and reactance design values, i.e., r_{ij}^d, x_{ij}^d for all $i, j \in \mathcal{N}$.
4: Observed data $(\mathbf{p}(t), \mathbf{q}(t), \mathbf{v}(t))$ for all $\mathcal{S}$.
5: Compute *incidence matrix* $\bar{\mathbf{A}}$ and *reduced incidence matrix* $\mathbf{A}$ of the network $\mathcal{T}$.
6: **procedure** ▷ (*State & Parameter estimation algorithm*)
7: Solve the following optimization problem and obtain $(\hat{\mathbf{r}}, \hat{\mathbf{x}}, \hat{\mathbf{v}}) \leftarrow \Omega(\mathbf{p}, \mathbf{q}, \mathbf{v})$

$$\min_{\hat{\mathbf{r}}, \hat{\mathbf{x}}, \hat{\mathbf{v}}} \quad \sum_{|\mathcal{S}|} ||\mathbf{v} - \hat{\mathbf{v}}||^2 \tag{7a}$$

$$\text{subject to} \quad \hat{\mathbf{v}} - \mathbf{v}_0 \mathbf{1}_n = \hat{\mathfrak{B}}\, \mathbf{q} + \hat{\mathfrak{D}}\, \mathbf{p} \tag{7b}$$

$$\hat{\mathfrak{B}} = 2\mathbf{A}^{-1} \left(\mathbf{I} + \hat{\Phi}\hat{\Psi}\right)^{-1} \mathcal{D}_{\hat{x}} \mathbf{A}^{-\top} \tag{7c}$$

$$\hat{\mathfrak{D}} = 2\mathbf{A}^{-1} \left(\mathbf{I} + \hat{\Phi}\hat{\Psi}\right)^{-1} \mathcal{D}_{\hat{r}} \mathbf{A}^{-\top} \tag{7d}$$

$$v_i v_j = \hat{\gamma}_{ij}\, \mathbf{v}_i + (1 - \hat{\gamma}_{ij})\, \mathbf{v}_j \quad \forall\, i, j \in \mathcal{N} \tag{7e}$$

$$\frac{\hat{r}_{ij}}{\hat{x}_{ij}} = \frac{r_{ij}^d}{x_{ij}^d}. \tag{7f}$$

8: Update $\hat{\mathbf{r}}$, $\hat{\mathbf{x}}$ and $\hat{\mathbf{v}}$. ▷ *Output data*

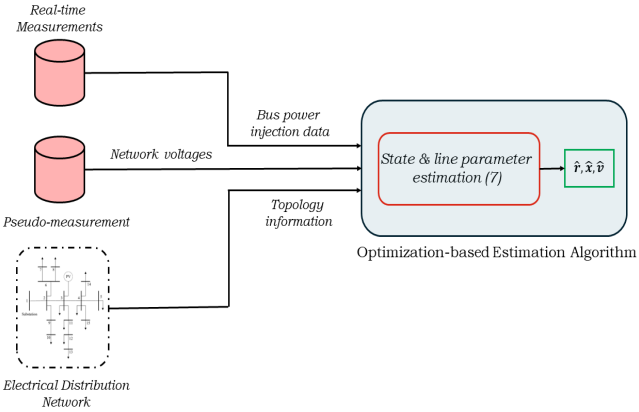


Fig. 1: Schematic diagram of the optimization-based state & parameter estimation algorithm.

Remark 3.1: To the best of our knowledge, this is the first instance, when an algorithmic solution is proposed to solve the state and parameter estimation problem of an electrical distribution system. We have witnessed algorithms that solve similar problems for electrical transmission systems, see [13], [14] and the reference therein. However, invoking these results to distribution systems are not straightforward due to several reasons: a) due to insufficient instrumentation, distribution systems suffer from the scarcity of real-time measurements; b) unlike transmission systems, distribution systems include numerous buses, thus the above algorithms raise scalability issues; c) distribution systems have high line resistance. Therefore, model simplification for transmission systems by ignoring line resistance becomes inaccurate for distribution systems. Note that, we attempt to address the above issues in Algorithm 1, while solving the joint estimation problem.

IV. Analysis of Algorithm

This section theoretically analyses the convergence properties of the state and the parameter estimation errors obtained via Algorithm 1. To facilitate this analysis, hereinafter we assume that the hyper-parameter γ is a constant. Although, in practice, this assumption is not required while implementing Algorithm 1. Considering this, the following proposition provides a sufficient condition to establish bounds on the parameter estimation error.

Proposition 1: Let $\mathbf{r} \in [r_{\min}, r_{\max}] \subseteq \mathbb{R}^n$ and $\mathbf{x} \in [x_{\min}, x_{\max}] \subseteq \mathbb{R}^n$ be the actual network resistance and reactance of an underlying distribution network, while the vectors $(\hat{\mathbf{r}}, \hat{\mathbf{x}}) \in \mathbb{R}^{2n}$ denote their estimates obtained via Algorithm 1. If they hold $\left|\left|\hat{\mathbf{r}} - \mathbf{r}\right|\right| \leq \alpha_1$ and $\left|\left|\hat{\mathbf{x}} - \mathbf{x}\right|\right| \leq \alpha_2$ for some $\alpha_1, \alpha_2 \in [0, \infty)$, then it leads to

$$\left|\left|\hat{\mathfrak{B}} - \mathfrak{B}\right|\right| \leq \alpha_{\mathfrak{B}}, \quad \left|\left|\hat{\mathfrak{D}} - \mathfrak{D}\right|\right| \leq \alpha_{\mathfrak{D}},$$

where $\alpha_{\mathfrak{B}}$ and $\alpha_{\mathfrak{D}}$ are the functions of $\alpha_1, \alpha_2, r_{\min}$ and $x_{\min}$. Since $\left|\left|\hat{\mathbf{r}} - \mathbf{r}\right|\right| \leq \alpha_1$ and $\left|\left|\hat{\mathbf{x}} - \mathbf{x}\right|\right| \leq \alpha_2$, the above proposition can be proved exploiting the definition of $\mathfrak{B}$ and $\mathfrak{D}$ from (5a) and (5b). Proposition 1 implies that if the optimization paradigm (7) yields bounded line parameter

estimates of $(\mathbf{r},\, \mathbf{x})$, then the estimated drift matrices obtained via Algorithm 1, are also bounded.

Next we attempt to investigate the convergence property of the voltage estimation error obtained via Algorithm 1. For this verification, first, we formulate voltage dynamics from the steady state equation (4). Let $\mathbf{v}_{\text{nom}}$ be the nominal voltage of the network and we define $\mathbf{e}(t) = \mathbf{v}(t) - \mathbf{v}_{\text{nom}}$, then differentiating both sides of (4), we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{e} = \underbrace{\begin{bmatrix}\mathfrak{B} & \mathfrak{D}\end{bmatrix}}_{\mathcal{H}} \underbrace{\begin{bmatrix}\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{q}^\top & \frac{\mathrm{d}}{\mathrm{dt}}\mathbf{p}^\top\end{bmatrix}^\top}_{\mathbf{u}}. \tag{8}$$

Eqn. (8) is derived to analyze the voltage estimation error. The aforesaid voltage dynamics has recently been explored to solve the voltage regulation problem for the class of distribution networks with high resistance to reactance ratio [3]. Because, in these networks, the reactive power can not solely regulate the voltage, and they require both active and reactive power set-points. In distribution networks, active and reactive power control loops for voltage regulation operate at fast time-scales. Therefore, unlike the traditional discrete-time model, we formulate (4) as a continuous-time dynamical system [15]. Observe that, since $\mathbf{v}(t)$ (thus $\mathbf{e}(t)$) and $\mathcal{H}$ are unknown in (8), this model can be reformulated as a perturbed system in terms of $\hat{\mathbf{v}}(t)$ (or say $\hat{\mathbf{e}}(t)$) and $\hat{\mathcal{H}}$ as

$$\frac{\mathrm{d}}{\mathrm{dt}}\hat{\mathbf{e}} = \hat{\mathcal{H}}\,\mathbf{u} + \left(\mathcal{H} - \hat{\mathcal{H}}\right)\mathbf{u}. \tag{9}$$

Hereinafter, we present results from *control lyapunov function* (CLF) and *exponential stability theorem*, which underpin the key theoretical analysis:

Definition 4.1: Consider the following nonlinear system

$$\frac{\mathrm{d}}{\mathrm{dt}}x(t) = f(x(t)) + g(x(t))\,u(t), \tag{10}$$

where $x(t) \in \mathcal{X}$, and $u(t) \in \mathcal{U}$ are the state and the control input, and the functions $f : \mathcal{X} \to \mathbb{R}^n$ and $g : \mathcal{X} \to \mathbb{R}^{n\times n}$ are locally lipschitz. Then a continuously differentiable function $V : \mathcal{X} \to \mathbb{R}_+$ is said to be a CLF for (10), if there exist (strictly) positive constants λ_1, λ_2, λ_3 and λ_4, such that

$$\lambda_1 \left|\left|x\right|\right|^2 \le V(x) \le \lambda_2 \left|\left|x\right|\right|^2, \tag{11a}$$

$$\left|\left|\nabla V(x)\right|\right| \le \lambda_3 \left|\left|x\right|\right| \tag{11b}$$

$$\exists\, u \in \mathcal{U} \text{ s.t } \nabla V(x)^\top (f(x) + g(x)u) \le -\lambda_4 \left|\left|x\right|\right|^2. \tag{11c}$$

Here $\nabla V(\cdot)$ is the gradient of $V(\cdot)$. The existence of a CLF implies that there exist an admissible, locally lipschitz, continuous state-feedback control update $\mathbf{k} : \mathcal{X} \to \mathcal{U}$, that globally asymptotically stable (10) to the origin. Furthermore, since $V(\cdot)$ is continuously differentiable, (11b) holds over any compact domain.

Definition 4.2: Let the origin be the equilibrium state of (10). Then, for $u(t) \equiv 0$, this equilibrium state is exponentially stable, if it is Lyapunov stable and there exist two positive constants κ, $\xi > 0$, such that

$$\left|\left|x(t)\right|\right| \le \kappa \exp^{-\xi(t-t_0)} \left|\left|x(t_0)\right|\right|. \tag{12}$$

Before examining the state estimation error, we adopt the subsequent assumption:

Assumption 1: There exist a feedback control $\mathbf{k} : [v_{\min},\, v_{\max}] \to [u_{\min},\, u_{\max}]$ for (9), such that it satisfies the linear growth bound, i.e., $\left|\left|\mathbf{k}(\hat{\mathbf{e}})\right|\right| \le \mu \left|\left|\hat{\mathbf{e}}\right|\right|$, for some $\mu \ge 0$.

Note that, since $\mathbf{k}(\cdot)$ is locally lipschitz, the above assumption is satisfied by definition. With this background, we state our results below:

Proposition 2: a) Let $V_1(\mathbf{e})$ be the CLF for the system (8). Then, the solution of (8) denoted as $\mathbf{e}(t)$, obeys (12) with $\kappa = \sqrt{\frac{\lambda_2}{\lambda_1}}$ and $\xi = \frac{\lambda_4}{2\,\lambda_1}$.

b) Let $\alpha_{\mathcal{H}} := \alpha_{\mathfrak{B}} + \alpha_{\mathfrak{D}}$, where $\alpha_{\mathfrak{B}}$ and $\alpha_{\mathfrak{D}}$ are defined in Proposition 1. Suppose $V_2(\hat{\mathbf{e}})$ is the CLF for (9), and its associated admissible controller $\mathbf{k}(\hat{\mathbf{e}})$ obeys Assumption 1. If the bound μ in Assumption 1, satisfies $\mu < \frac{\lambda_4}{\lambda_3\,\alpha_{\mathcal{H}}}$, then $\hat{\mathbf{e}}(t)$ solves as (12), where $\kappa = \sqrt{\frac{\lambda_2}{\lambda_1}}$ and $\xi = \frac{\lambda_4 - \alpha_{\mathcal{H}}\mu\lambda_3}{2\,\lambda_1}$.

Proof: a) Differentiating $V_1(\mathbf{e})$ with respect to (8), we find $\frac{\mathrm{d}}{\mathrm{dt}}V_1(\mathbf{e}) = \nabla V_1(\mathbf{e})^\top\, (\mathcal{H}\mathbf{u})$. Since $V_1(\mathbf{e})$ is the CLF for (8), leveraging (11c) and (11a), one gets $\dot{V}_1(\mathbf{e}) \le -\lambda_4 \left|\left|\mathbf{e}\right|\right|^2 \le -\frac{\lambda_4}{\lambda_1} V_1(\mathbf{e})$. This simplifies as (12), by doing algebraic manipulation.

b) Compute $\dot{V}_2(\hat{\mathbf{e}})$ with respect to (9), and find

$$\begin{aligned}\dot{V}_2(\hat{\mathbf{e}}) &= \nabla V_2(\hat{\mathbf{e}})^\top \left(\hat{\mathcal{H}}\,\mathbf{u} + \left(\mathcal{H} - \hat{\mathcal{H}}\right)\mathbf{u}\right) \\ &\le -\left(\lambda_4 - \alpha_{\mathcal{H}}\mu\lambda_3\right) \left|\left|\hat{\mathbf{e}}\right|\right|^2 \le -\frac{\lambda_4 - \alpha_{\mathcal{H}}\mu\lambda_3}{\lambda_1} V_2(\hat{\mathbf{e}}).\end{aligned}$$

Here the first inequality is obtained employing Proposition 1, Definition 4.1 and Assumption 1, while the second inequality arises due to (11a). Since we select $\mu < \frac{\lambda_4}{\lambda_3\,\alpha_{\mathcal{H}}}$, the aforesaid inequality reduces to (12). ■

Proposition 3: The estimated voltage profile $\hat{\mathbf{v}}(t)$ converges to the ground truth voltage $\mathbf{v}(t)$, exponentially.

Proof: First, we observe

$$\begin{aligned}\left|\left|\hat{\mathbf{v}}(t) - \mathbf{v}(t)\right|\right| &\le \left|\left|\mathbf{v}(t) - \mathbf{v}_{\text{nom}}\right|\right| + \left|\left|\hat{\mathbf{v}}(t) - \mathbf{v}_{\text{nom}}\right|\right| \\ &=: \left|\left|\mathbf{e}(t)\right|\right| + \left|\left|\hat{\mathbf{e}}(t)\right|\right|.\end{aligned}$$

Then, employing Proposition 2, we find that $(\hat{\mathbf{v}}(t) - \mathbf{v}(t))$ satisfies (12), where $\xi = \lambda_4/\lambda_1$. Since both these quantities are strictly positive constants, $\hat{\mathbf{v}}(t)$ converges to $\mathbf{v}(t)$ exponentially, which completes the proof. ■

Proposition 2 and 3 explicitly establish that if one obtains bounded network impedance values via Algorithm 1, then the estimated voltage profiles obtained via the same algorithm, exponentially converge to the ground truth voltage.

V. Experimental set-up and evaluation

We evaluate our proposed method on the IEEE 33 bus distribution network using real-world load and solar generation profiles. In this network, solar PV panels of 4 MW capacity are installed at buses 2, 5, 12, and 18 (see Fig. 2), and the solar generation data of these PV panels are obtained from [16] for a specific simulation month. Concurrently, we gather active and reactive power injection data from prosumers,

termed as base load data $(\mathbf{p}_{\text{base}}, \mathbf{q}_{\text{base}})$, sampled hourly. The design values for $\mathcal{D}_{\mathbf{r}}$ and $\mathcal{D}_{\mathbf{x}}$ are sourced from [17]. Our experiments aim to gauge the lossy model's accuracy compared to the lossless model in estimating voltage and line parameters ($\mathbf{r}$ and $\mathbf{x}$).

Measurements: We employ pandapower [17] to solve the true nonlinear power flow equations of the distribution network using the base load data $(\mathbf{p}_{\text{base}}, \mathbf{q}_{\text{base}})$. This input data is taken as the real-world $(\mathbf{p}, \mathbf{q})$ injection samples at the buses and the output bus voltages $\mathbf{v}$ are taken as the observed real-world voltage measurements.

Baseline (lossless model): We run Algorithm 1 to estimate the voltage and line parameters from the real-world samples of $(\mathbf{p}, \mathbf{q})$ by setting $\gamma_{ij} = 0.5$ for all $i, j \in \mathcal{N}$.

Our approach (lossy model): We further use Algorithm 1 to estimate the voltage and line parameters from $(\mathbf{p}, \mathbf{q})$. However, here γ_{ij}s are also estimated using (3) based on measured voltage samples.

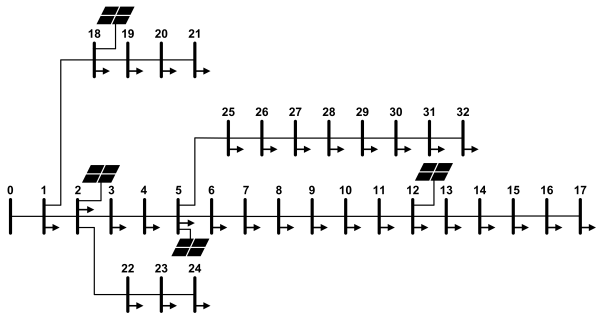


Fig. 2: Illustration of the IEEE 33 bus distribution network.

A. Model sensitivity to load data

For our experiments, we create high-load scenarios by scaling the base loads $\mathbf{q}_{\text{base}}$ with a factor ψ, where $\psi \in \{2, 3, 5\}$. We keep $\mathbf{p}_{\text{base}}$ fixed throughout and only scale $\mathbf{q}_{\text{base}}$, as variations in reactive power drive voltage magnitude fluctuations in the distribution network. We adopt the lossy distribution flow model from Section II for Algorithm 1, contrasting its performance with the traditional lossless model from [2, Eq. (3)]. We consider two scenarios: a) when the exact values of the network impedance are known, and b) when the network impedance values are estimated using Algorithm 1. Figure 3a shows that with known impedance values, the lossy model consistently outperforms the lossless model, with percentage errors in voltage prediction increasing notably with higher reactive loads—from 0.4% at base load to 4.1% at $\psi = 5$.

Figures 3b and 3c show the performance when the network impedance are unknown. These figures display percentage errors in voltage and $\mathbf{r}$ (and $\mathbf{x}$) predictions, respectively. Notably, Algorithm 1 yields lower voltage prediction errors (Figure 3b) compared to errors when true $\mathbf{r}$ and $\mathbf{x}$ are known (Figure 3a). Algorithm 1 minimizes the sum of squared differences between predicted and observed voltages, potentially deviating $\mathbf{r}$ and $\mathbf{x}$ from actual values (Figure 3c). Despite both models using Algorithm 1, the proposed lossy model considerably reduces the voltage estimation error, especially under higher loads. In terms of estimating the line resistances and reactances, the proposed lossy model reduces the estimation error by more than 50% with respect to the lossless model across all loading conditions, thereby underscoring its superior performance. Predicting $\mathbf{r}$ and $\mathbf{x}$ values accurately can impact downstream tasks like estimating line current and line power flows, which extends beyond the scope of our current study.

In Fig. 4 and 5, we further consider the base load data and compute the evolution of the estimated and the measured voltage profiles for the network buses, which are adjacent to the PV panels, i.e., bus 3, 6, 13 and 19. We observe that the estimation accuracy of the lossy model is much better compare to the lossless model, as expected.

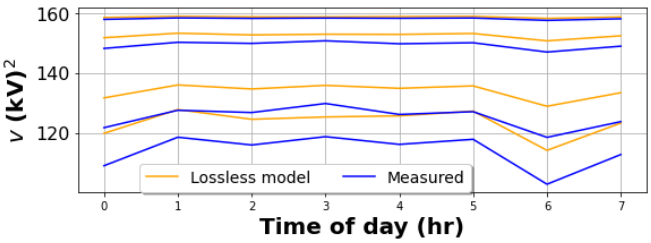


Fig. 4: Estimated and measured voltage of the lossless distribution system for the bus 3, 6, 13, 19.

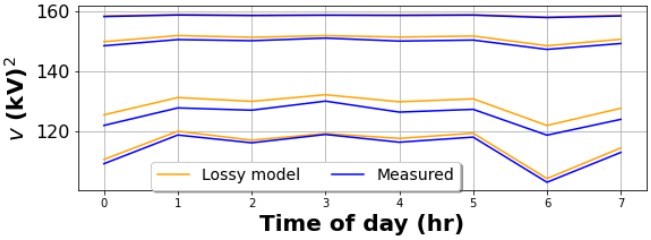


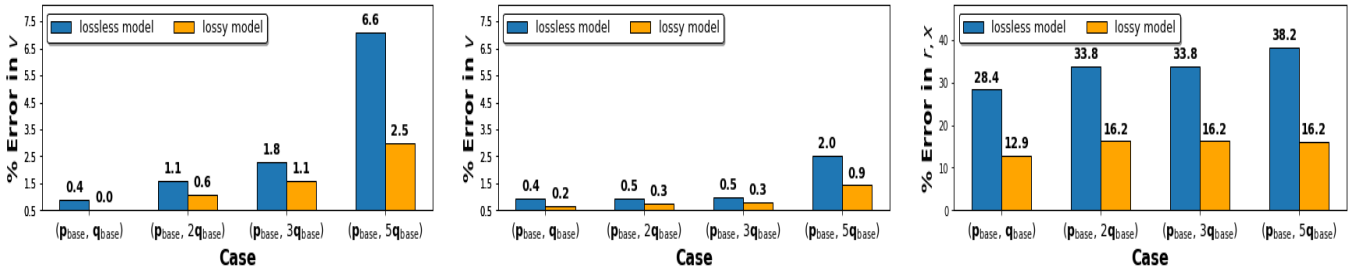
Fig. 5: Estimated and measured voltage of the lossy distribution system for the bus 3, 6, 13, 19.

B. Model sensitivity to solar injection

We further analyze the sensitivity of bus voltage and network impedance estimation for various solar generation profiles. We, in particular, examine high-load conditions $2\mathbf{q}_{\text{base}}$ and $3\mathbf{q}_{\text{base}}$. Three distinct solar generation profiles are synthesized: a) no solar injection ($\varepsilon = 0$), b) peak solar injection ($\varepsilon = 1$), and c) reduced solar injection ($\varepsilon = 0.25$), achieved by setting the solar capacity of each PV panel to 1 MW (see Fig. 2). Considering this, the voltage and parameter estimation errors are depicted in Fig. 6 and Fig. 7. From these illustrations, we learn that the lossy model outperforms the lossless model across all cases, providing more accurate predictions of both bus voltages and network impedance.

VI. Concluding Remarks

In this paper we propose an optimization-based algorithm, which considers the active and the reactive power bus injection and network voltage data as ground truths and generates



(a) % error in voltage prediction with known $\mathbf{r}$ and $\mathbf{x}$

(b) % error in voltage prediction with estimated $\mathbf{r}$ and $\mathbf{x}$

(c) Average total % error in estimating $\mathbf{r}$ and $\mathbf{x}$ across lines

Fig. 3: Voltage and line parameter percentage errors obtained via Algorithm 1 for different grid loading conditions.

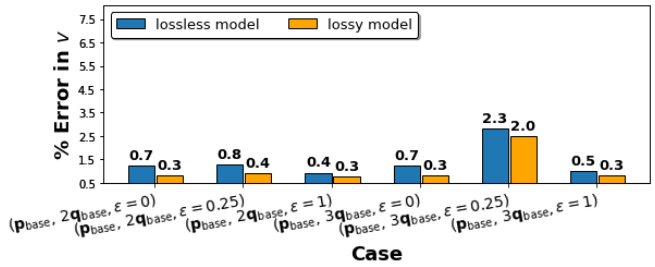


Fig. 6: % error in voltage prediction with estimated values of $\mathbf{r}$ and $\mathbf{x}$ (using Algorithm 1)

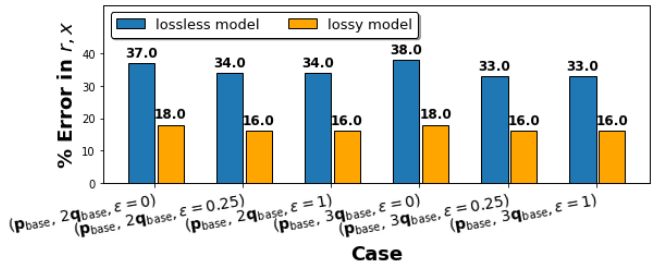


Fig. 7: Average total % error in estimating $\mathbf{r}$ and $\mathbf{x}$ across lines

voltage and impedance estimates for a single-phased radial power distribution network. We adopt the lossy distribution flow equation to model the distribution network, which yields more accurate state and impedance estimates than the one obtained using the traditional lossless models. We leverage tools from the control Lyapunov function and perturbation theory, and show that, if the parameter estimation error is bounded, then the estimated voltage profile exponentially converges to the original voltage data. We simulate this algorithm on the standard IEEE 33-bus distribution network considering the lossy distribution flow model. We find that the proposed approach considerably improves the impedance estimation at all loading conditions, while reasonably improving the voltage estimation at higher loads.

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