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Cohesive-Tracking versus Fast-Convergence with Dynamic Average Consensus

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Abstract— **This work investigates the tracking of dynamic trajectories using consensus-based decentralized algorithms. Dynamic average consensus (DAC) algorithms seek to improve tracking of the average signal by increasing the convergence rate, e.g., using accelerated versions of standard DAC algorithms. The main contribution of this work is to show that improving the convergence rate doesn't necessarily ensure improved tracking of the average signal, especially during rapid changes in the measured signals. Additionally, this work also presents a DAC-based delayed self reinforcement (DSR) approach to improve the tracking of the average signal rather than the convergence rate. The proposed DSR-DAC algorithm, which approximates the ideal centralized DAC, improves tracking of the average signal by only using already-available (current and past), local information from the network and sensed reference signals. Numerical simulation results for a multi-mobile agent tracking problem demonstrate substantial improvement with 91% and 84% reduction in tracking error using the proposed DSR-DAC approach when compared to the standard and accelerated DAC algorithms, respectively.**

I. Introduction

The dynamic average consensus (DAC) problem involves a network of agents working collectively to measure and track the time average of their individual time-varying reference signals. Unlike the traditional static goal of achieving agreement on a fixed (time-invariant) average value, DAC seeks to use the Laplacian-based consensus algorithm [1] to track a dynamic (time-varying) average signal. This problem of tracking the time-varying average is relevant to applications such as: sensor fusion with distributed sensor networks [2], formation control in leaderless multi-robot coordination [3], gradient tracking in distributed optimization [4], and demand-response averaging in distributed resource allocation [5]. In all these applications, it is important that the agents achieve and maintain consensus on the average at all times and not just at the steady state value. Therefore, this work seeks to ensure that all agents maintain consensus on the average at all times and not just at the steady state, with all agents having the same time-varying average value.

The use of DAC to track the average of each agent's time-varying reference signal, with zero-steady state tracking error, has been well studied in literature [6], [7], and a survey of the algorithms is provided in [8]. A challenge with DAC algorithms is that the tracking error can be significant when the reference signals are rapidly changing, especially due to communication bandwidth constraints with discrete-time implementation. This has motivated the development of accelerated DAC algorithms that used delayed information from the network. For example, previous works have shown that the use of delayed information can improve the convergence rate in the static (time-invariant) consensus [9] as well as the time-varying DAC [10]. Although improving the convergence rate with the accelerated DAC leads to more cohesive tracking compared to the standard DAC approach, the discrete-time accelerated DAC does not necessarily ensure cohesive tracking of the average signal with rapidly varying reference signals, which is addressed in this article.

This article investigates DAC algorithms for cohesive tracking of the time-average of the reference signals rather than fast convergence rate. Specifically, a delayed self reinforcement (DSR) method for designing a DAC algorithm with cohesive tracking of the time-average of the reference signals is proposed. Different from the existing works that use the DSR approach to design leader-follower consensus algorithms where only a few leader agents have the source information as in [11], this work develops a DSR-based DAC algorithm to track the time-average of each agent's reference signal. The proposed DSR-DAC approach uses only already-available current and past information from the network and reference signals and does not require an increase in the communication bandwidth between the agents or changing the communication network structure. The proposed DSR-DAC approximates the ideal (centralized) cohesive DAC (where each agent has access to the average information), and therefore, the DSR-DAC significantly improves the cohesive tracking of rapidly changing average signal when compared to the existing DAC and accelerated DAC algorithms. In addition to developing the new DSR-DAC, comparative simulation results are provided to demonstrate the improvements in tracking the time-varying average signal with and without the proposed approach. Finally, simulation results indicate that designing the parameters for DAC algorithms to achieve cohesive tracking of the average requires different parameters than those needed for the fastest convergence rate.

II. Background and Problem Setup

Consider a system with N agents (nodes) in a communication network. The communication network of N agents is represented as a connected, undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \ldots, N\}$ is the node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set representing communication between agents. The adjacency matrix $\mathbf{A} = [a_{ij}]$ of the graph $\mathcal{G}$ is defined as a $\mathbf{R}^{N \times N}$ matrix where the entry $a_{ij} = 1$ if agent i can send information to agent and j (i.e., $(i, j) \in \mathcal{E}$), and 0 otherwise. The Laplacian matrix $\mathbf{L}$ is defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ is the degree matrix, a diagonal matrix with $d_{ii} = \sum_{j=1}^{N} a_{ij}$ representing the degree of node i, i.e., the number of edges connected to node i. For a connected undirected graph $\mathcal{G}$,

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the Laplacian matrix satisfies [10]

$$\mathbf{L}\mathbf{1}_N = \mathbf{0}_N, \tag{1}$$

where $\mathbf{0}_N$ and $\mathbf{1}_N$ is the vector of ones and zeros of size N, respectively. Moreover, the $\mathbf{L}$ has a simple eigenvalue $\lambda_{\mathbf{L},1} = 0$ and all other eigenvalues satisfy $\lambda_{\mathbf{L},1} = 0 < \lambda_{\mathbf{L},2} = \underline{\lambda}_{\mathbf{L}} \leq \cdots \leq \lambda_{\mathbf{L},N} = \overline{\lambda}_{\mathbf{L}}$.

A. Dynamic average consensus (DAC)

In the dynamic average consensus (DAC) problem, each agent i has a locally measurable reference signal $u_i(t) : [0,\infty) \to \mathbb{R}$ and aims to track the time-varying average $u_{\text{avg}}(t)$ of the reference signals of all agents, i.e.,

$$u_{\text{avg}}(t) = \frac{1}{N}\sum_{i=1}^{N} u_i(t), \tag{2}$$

using the following distributed algorithm [6],

$$\dot{x}_i(t) = \alpha \sum_{i=1}^{N} a_{ij}\left(x_j(t) - x_i(t)\right) + \dot{u}_i(t), \tag{3a}$$

$$x_i(0) = u_i(0), \quad \forall i \in \mathcal{V}, \tag{3b}$$

where $\alpha > 0$ is an update gain, $x_i(t)$ is the state of agent i, and $\dot{u}_i(t)$ is the time derivative of each reference signal $u_i(t)$. The tracking error $e_i(t)$ for each agent is the deviation of its state $x_i(t)$ from the average $u_{\text{avg}}(t)$ signal,

$$e_i(t) = x_i(t) - u_{\text{avg}}(t). \tag{4}$$

Compactly, the DAC algorithm in Eq. (3) can be rewritten for all agents, as,

$$\dot{\mathbf{x}}(t) = -\alpha\mathbf{L}\mathbf{x}(t) + \dot{\mathbf{u}}(t), \tag{5a}$$

$$\mathbf{x}(0) = \mathbf{u}(0), \tag{5b}$$

where $\mathbf{x}(t) = (x_1(t), \cdots, x_N(t))^T$ and $\mathbf{u}(t) = (u_1(t), \cdots, u_N(t))^T$. The accelerated version of Eq. (5) for improving the convergence rate using outdated feedback, is given in [10]

$$\dot{\mathbf{x}}(t) = -\alpha\left(1-\beta\right)\mathbf{L}\mathbf{x}(t) - \alpha\beta\mathbf{L}\mathbf{x}(t-\tau) + \dot{\mathbf{u}}(t), \tag{6a}$$

$$\mathbf{x}(0) = \mathbf{u}(0), \mathbf{x}(\bar{\tau}) = 0 \text{ for } \bar{\tau} \in [-\tau, 0), \tag{6b}$$

where $\beta \in \mathbf{R}$ is a split factor and $\tau \geq 0$ is a delay term. When $\beta = 0$, Eq. (6) becomes the standard DAC algorithm in Eq. (5).

A challenge is that the algorithm in Eq. (5) (and similarly, in Eq. (6)) requires explicit knowledge of the derivative of the reference signals, which might not always be available [8]. Additionally, the initial condition requirement $\mathbf{x}(0) = \mathbf{u}(0)$ may not always be met, and perturbations in the initial condition $\mathbf{u}(0)$ results in a steady-state tracking error. These issues can be removed by the following change of variables

$$\mathbf{q}(t) = \mathbf{x}(t) - \mathbf{u}(t), \tag{7}$$

which can be substituted back to Eq. (6) to obtain the following accelerated DAC method,

$$\begin{aligned}\dot{\mathbf{q}}(t) = &-\alpha\mathbf{L}\left(\mathbf{q}(t) + \mathbf{u}(t)\right) + \alpha\beta\mathbf{L}\left(\mathbf{u}(t) - \mathbf{u}(t-\tau)\right)\\ &+ \alpha\beta\mathbf{L}\left(\mathbf{q}(t) - \mathbf{q}(t-\tau)\right),\end{aligned} \tag{8a}$$

$$\mathbf{x}(t) = \mathbf{q}(t) + \mathbf{u}(t), \tag{8b}$$

$$\mathbf{q}(\bar{\tau}) = \mathbf{0}_N \text{ for } \bar{\tau} \in [-\tau, 0]. \tag{8c}$$

The convergence rate of the continuous time DAC with continuous communication can be increased arbitrarily by increasing the gain α. However, this assumes that the sensing and communication bandwidth is infinite. To account for communication limitations, discrete-time versions of the DAC algorithms are considered throughout the rest of the article. In particular, the accelerated DAC algorithm in Eq. (8) can be discretized with $\mathbf{x}[k] = \mathbf{x}(t)|_{t=k\Delta t}$, as

$$\begin{aligned}\mathbf{q}[k+1] = &\,\mathbf{q}[k] - \alpha\Delta t\mathbf{L}\left(\mathbf{q}[k] + \mathbf{u}[k]\right)\\ &+ \alpha\beta\Delta t\mathbf{L}\left(\mathbf{u}[k] - \mathbf{u}[k-1]\right)\\ &+ \alpha\beta\Delta t\mathbf{L}\left(\mathbf{q}[k] - \mathbf{q}[k-1]\right),\end{aligned} \tag{9a}$$

$$\mathbf{x}[k] = \mathbf{q}[k] + \mathbf{u}[k], \tag{9b}$$

$$\mathbf{q}[-1] = \mathbf{q}[0] = \mathbf{0}_N, \tag{9c}$$

where k represents a discrete-time instance, and the delay τ is selected as the sampling period, i.e., $\tau = \Delta t$. The accelerated DAC can be rewritten by consolidating the parameters in Eq. (9a), as

$$\begin{aligned}\mathbf{q}[k+1] = &\,\mathbf{q}[k] - \gamma\mathbf{L}\left(\mathbf{q}[k] + \mathbf{u}[k]\right)\\ &+ \beta_1\mathbf{L}\left(\mathbf{u}[k] - \mathbf{u}[k-1]\right)\\ &+ \beta_1\mathbf{L}\left(\mathbf{q}[k] - \mathbf{q}[k-1]\right).\end{aligned} \tag{10}$$

where $\gamma = \alpha\Delta t > 0$ and $\beta_1 = \alpha\beta\Delta t$ are the new scalar parameters.

B. Stability of accelerated DAC

The stability requirement on the accelerated DAC leads to conditions on the gain selection (γ, β_1), which can be found using the characteristic equation of Eq. (10). For undirected graphs, the Laplacian $\mathbf{L}$ is symmetric with real, non-negative eigenvalues $\lambda_{\mathbf{L},m}$. Then, the characteristic equation of Eq. (10) for each eigenvalue $\lambda_{\mathbf{L},m}$ is

$$\phi(z) = z^2 + \left(-1 + \gamma\lambda_{\mathbf{L},m} - \beta_1\lambda_{\mathbf{L},m}\right)z + \beta_1\lambda_{\mathbf{L},m}. \tag{11}$$

For $\lambda_{\mathbf{L},1} = 0$, the two roots of Eq. (11) are at $z = 1$, corresponding to the consensus direction, i.e., the $\mathbf{1}_N$ direction, and at $z = 0$.

For $\lambda_{\mathbf{L},m} > 0$, stability conditions from the Jury test are

$$\phi(1) = \gamma\lambda_{\mathbf{L},m} > 0, \tag{12a}$$

$$(-1)^2\phi(-1) = 2 - \gamma\lambda_{\mathbf{L},m} + 2\beta_1\lambda_{\mathbf{L},m} > 0, \tag{12b}$$

$$|\beta_1\lambda_{\mathbf{L},m}| < 1, \tag{12c}$$

which give the following bounds for (γ, β_1),

$$0 < \gamma, \quad \frac{\gamma}{2} - \frac{1}{\overline{\lambda}_{\mathbf{L}}} \overset{(12b)}{<} \beta_1 \overset{(12c)}{<} \frac{1}{\overline{\lambda}_{\mathbf{L}}}, \tag{13}$$

where the bracketed numbers on inequalities indicate which condition/equation is used.

For the standard DAC, i.e., $\beta_1 = 0$ in Eq. (10), condition (12b) gives the following stability upper bound for γ,

$$\gamma < \frac{2}{\overline{\lambda}_{\mathbf{L}}}. \tag{14}$$

C. Optimal convergence parameter selection

For the standard DAC, the characteristic equation has a single pole z_p for each eigenvalue $\lambda_{\mathbf{L},m}$, i.e.,

$$z_p = 1 - \gamma\lambda_{\mathbf{L},m}. \tag{15}$$

Thus, to optimize the convergence rate, the parameter γ is chosen such that all the poles are farthest away from the unit circle. In other words, all the poles are inside a circle with the minimal radius ρ centered at the origin. The optimal parameter γ^* is given by [8],

$$\gamma^* = \frac{2}{\overline{\lambda}_{\mathbf{L}} + \underline{\lambda}_{\mathbf{L}}}. \tag{16}$$

III. Proposed Approach

A. DSR-DAC derivation

The proposed DSR-DAC is derived from the following ideal centralized approach where each agent i has access to the average signal

$$\dot{\mathbf{x}}(t) = -\alpha\mathbf{x}(t) + \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T\left(\alpha\mathbf{u}(t) + \dot{\mathbf{u}}(t)\right). \tag{17}$$

The centralized algorithm in Eq. (17) exponentially tracks the average of the reference signals $\mathbf{u}(t)$ provided $\mathbf{u}(t)$ is differentiable since the resulting tracking error (Eq. (4)), for each agent i,

$$\dot{e}_i(t) = -\alpha e_i(t), \tag{18}$$

exponentially converges to zero, where the average is given by $u_{\text{avg}}(t) = \frac{1}{N}\mathbf{1}_N^T\mathbf{u}(t)$. Thus, each agent in the ideal centralized algorithm has the same response, i.e., is cohesive and exponentially tracks the average signal. The ideal centralized algorithm in Eq. (17) can be decentralized by multiplying both sides of Eq. (17) by $\beta\mathbf{L}$ and applying $\mathbf{L}\mathbf{1}_N = \mathbf{0}_N$ from Eq. (1) to obtain

$$\beta\mathbf{L}\dot{\mathbf{x}}(t) = -\alpha\beta\mathbf{L}\mathbf{x}(t). \tag{19}$$

Adding $\dot{\mathbf{x}}(t)$ to both sides of Eq. (19) and rearranging terms leads to

$$\dot{\mathbf{x}}(t) = -\alpha\beta\mathbf{L}\mathbf{x}(t) + (\mathbf{I} - \beta\mathbf{L})\,\dot{\mathbf{x}}(t), \tag{20}$$

where $\mathbf{I}$ is the identity matrix with dimension $N \times N$. Substituting the change of variables from Eq. (7) into Eq. (20) and rearranging terms, results in

$$\dot{\mathbf{q}}(t)+\dot{\mathbf{u}}(t) = -\alpha\beta\mathbf{L}\left(\mathbf{q}(t) + \mathbf{u}(t)\right)+(\mathbf{I} - \beta\mathbf{L})\left(\dot{\mathbf{q}}(t) + \dot{\mathbf{u}}(t)\right). \tag{21}$$

Then, using a delay-based derivative to approximate the derivatives on the right hand side of Eq. (21), with delay τ, the continuous-time, cohesive DSR-DAC update law becomes

$$\begin{aligned}\dot{\mathbf{q}}(t) = &-\alpha\beta\mathbf{L}\left(\mathbf{q}(t) + \mathbf{u}(t)\right) - \beta\mathbf{L}\left(\frac{\mathbf{u}(t) - \mathbf{u}(t-\tau)}{\tau}\right)\\ &+ (\mathbf{I} - \beta\mathbf{L})\left(\frac{\mathbf{q}(t) - \mathbf{q}(t-\tau)}{\tau}\right),\end{aligned} \tag{22a}$$

$$\mathbf{x}(t) = \mathbf{q}(t) + \mathbf{u}(t), \tag{22b}$$

$$\mathbf{q}(\bar{\tau}) = \mathbf{0}_N \text{ for } \bar{\tau} \in [-\tau, 0]. \tag{22c}$$

Selecting the sampling time period Δt as the delay τ (the smallest possible one), the cohesive DSR-DAC update law can be discretized as,

$$\begin{aligned}\mathbf{q}[k+1] = &\,\mathbf{q}[k] - \alpha\beta\Delta t\mathbf{L}\left(\mathbf{q}[k] + \mathbf{u}[k]\right)\\ &- \beta\mathbf{L}\left(\mathbf{u}[k] - \mathbf{u}[k-1]\right)\\ &+ (\mathbf{I} - \beta\mathbf{L})\left(\mathbf{q}[k] - \mathbf{q}[k-1]\right),\end{aligned} \tag{23a}$$

$$\mathbf{x}[k] = \mathbf{q}[k] + \mathbf{u}[k], \tag{23b}$$

$$\mathbf{q}[-1] = \mathbf{q}[0] = \mathbf{0}_N. \tag{23c}$$

The DSR-DAC update can be considered as a special case of the following generalized update,

$$\begin{aligned}\mathbf{q}[k+1] = &\,\mathbf{q}[k] - \gamma\mathbf{L}\left(\mathbf{q}[k] + \mathbf{u}[k]\right)\\ &+ \beta_1\mathbf{L}\left(\mathbf{u}[k] - \mathbf{u}[k-1]\right)\\ &+ \beta_1\mathbf{L}\left(\mathbf{q}[k] - \mathbf{q}[k-1]\right)\\ &+ \beta_2\left(\mathbf{q}[k] - \mathbf{q}[k-1]\right),\end{aligned} \tag{24}$$

when $\gamma = \alpha\beta\Delta t$, with $\beta_1 = -\beta$, and $\beta_2 = 1$.

Remark 1: Both methods, the accelerated DAC in Eq. (10) and the cohesive DSR-DAC in Eq. (24), use the outdated-feedback term $\beta_1\mathbf{L}\left(\mathbf{q}[k] - \mathbf{q}[k-1]\right)$. However, compared to the update law for the accelerated DAC, the cohesive DSR-DAC contains an additional momentum term, i.e., $\beta_2\left(\mathbf{q}[k] - \mathbf{q}[k-1]\right)$, which further improves cohesion.

B. Stability of DSR-DAC

The stability of the DSR-DAC is characterized by the gain selection $(\gamma, \beta_1, \beta_2)$ found using the characteristic equation of Eq. (24). For each $\lambda_{\mathbf{L},m}$, the characteristic equation is given by,

$$z^2 + \left(-1 - \beta_2 + \gamma\lambda_{\mathbf{L},m} - \beta_1\lambda_{\mathbf{L},m}\right)z + \left(\beta_2 + \beta_1\lambda_{\mathbf{L},m}\right). \tag{25}$$

For $\lambda_{\mathbf{L},1} = 0$, one of the roots of Eq. (25) is at $z = 1$ corresponding to the consensus direction, i.e., the $\mathbf{1}_N$ direction, while the other root is at $z = \beta_2$, which implies that for stability

$$-1 < \beta_2 \leq 1. \tag{26}$$

For $\lambda_{\mathbf{L},m} > 0$, the Jury test [12] gives the stability conditions for the DSR-DAC,

$$0 < \gamma, \quad \frac{\gamma}{2} - \frac{\beta_2 + 1}{\lambda_{\mathbf{L},m}} < \beta_1 < \frac{1 - \beta_2}{\lambda_{\mathbf{L},m}}, \tag{27}$$

and provided $0 < \beta_2 \leq 1$, then β_1 can be bounded by the largest Laplacian eigenvalue $\overline{\lambda}_{\mathbf{L}}$ as,

$$\frac{\gamma}{2} - \frac{\beta_2 + 1}{\overline{\lambda}_{\mathbf{L}}} < \beta_1 < \frac{1 - \beta_2}{\overline{\lambda}_{\mathbf{L}}}. \tag{28}$$

Remark 2: For undirected graphs with maximum degree $\overline{\Delta}$, a more conservative upper bound on β_1 in Eq. (28) can obtained without explicit knowledge of the eigenvalues of the Laplacian. From the Gershgorin circle theorem $\overline{\lambda}_{\mathbf{L}} \leq 2\overline{\Delta}$, which implies that β_1 can be bounded, using the maximum degree $\overline{\Delta}$, as

$$\frac{\gamma}{2} - \frac{\beta_2 + 1}{2\overline{\Delta}} < \beta_1 < \frac{1 - \beta_2}{2\overline{\Delta}}.$$

C. Parameter selection for fast convergence

The parameters $(\gamma, \beta_1, \beta_2)$ in the DSR-DAC method can be selected to optimize the convergence rate. For the fastest convergence rate, the parameters of the DSR-DAC are selected such that the roots of Eq. (25) are farthest away from the unit circle. This is achieved when the poles, associated with the nonzero extremal eigenvalues, i.e., $\underline{\lambda}_{\mathbf{L}}, \overline{\lambda}_{\mathbf{L}}$, are equidistant from the origin. The parameters for optimal convergence are [12],

$$\gamma = \frac{4}{\left(\sqrt{\overline{\lambda}_{\mathbf{L}}} + \sqrt{\underline{\lambda}_{\mathbf{L}}}\right)^2}, \quad \beta_1 = 0, \quad \beta_2 = \frac{\left(\sqrt{\overline{\lambda}_{\mathbf{L}}} - \sqrt{\underline{\lambda}_{\mathbf{L}}}\right)^2}{\left(\sqrt{\overline{\lambda}_{\mathbf{L}}} + \sqrt{\underline{\lambda}_{\mathbf{L}}}\right)^2}. \tag{29}$$

This shows that the DSR-DAC only relies on the momentum term for fast convergence and does not use the outdated-feedback term defined in Rem 1. This is referred to as the momentum DAC (M-DAC).

D. Parameter selection for accurate cohesive tracking

The DSR-DAC in Eq. (23) (with $\gamma = \alpha\beta\Delta t$, $\beta_1 = -\beta$ and $\beta_2 = 1$ in Eq. (24)) approximates the ideal DAC, i.e., Eq. (17), which has a settling time T_s value (for staying within 2% of a final value),

$$\alpha = 4/T_s. \tag{30}$$

The parameter α can be selected to achieve a specified settling time T_s, and the parameter β can be optimized within its stability bounds, i.e., from Eq. (28),

$$0 < \beta < \frac{4}{\overline{\lambda}_{\mathbf{L}}\left(\alpha\Delta t + 2\right)}. \tag{31}$$

IV. Simulation results and discussion

In this section, the tracking performance is compared between the accelerated DAC (Eq. (10)), standard DAC (Eq. (10) with $\beta_1 = 0$), momentum DAC (Eq. (24) with $\beta_1 = 0$) and DSR-DAC (Eq. (24)). The update gains for each method are chosen to minimize a tracking error metric for a set of reference signals. These signals correspond to the position signals along the horizontal x-direction for each agent in the network of mobile agents, shown in Fig. 1. The goal of the agents is to track the overall average of their individual time-varying positions, which, in this example, is $p^\star_x(t) = (1/N)\sum_{i=1}^{5} p_{ix}(t)$.

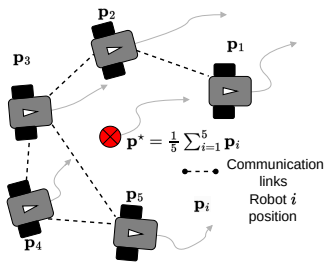


Fig. 1: Example network of $N = 5$ mobile agents with pose $\mathbf{p}_i$ for $i \in \{1, \cdots, N\}$ communicating over an undirected network topology. The agents use a dynamic average consensus algorithm to track the average of their pose $\mathbf{p}^\star$.

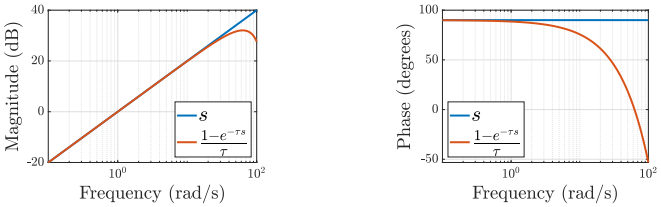
A. Parameter selection for simulations

The choice of parameters in the different DAC methods is obtained empirically by minimizing the following tracking error metric over a set of generated signals indexed by r,

$$\varepsilon = \| \max_{i,r} |\mathbf{e}_{i,r}[k]| \|_\infty, \tag{32}$$

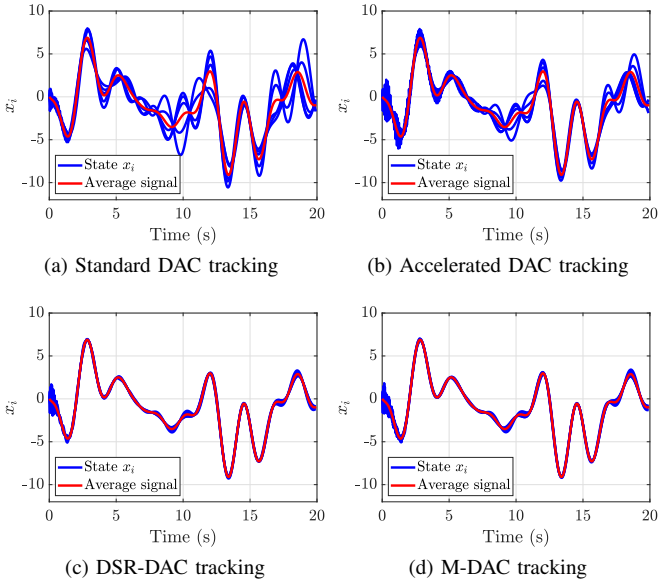
where $\|\cdot\|_\infty$ represents the ∞-norm, $|\cdot|$ is the component-wise absolute value, and $\mathbf{e}_{i,r} = (e_{i,r}[k_0], \cdots, e_{i,r}[k_H])^T$ is the sequence of errors in the tracking for agent i for $H = k_H - k_0$ time steps. The start of the evaluation k_0 is sufficiently large such that the effects from the initial transients have dissipated. The tracking error metric is minimized using the example network in Fig. 1. The sampling time for each algorithm is chosen to be $\Delta t = \tau = 0.05$ s, the start of error evaluation is kept the same at $k_0 = 2/\Delta t = 40$, and the final time step for simulations is $k_H = 20/\Delta t = 400$. For the sampling time $\Delta t = \tau = 0.05$ s, the bode plots for the magintude and phase variation of the delay-based derivative are shown in Fig. 2. The frequencies of the signals are chosen to be less than a max frequency ω^* where the gain and phase of the delay-based approximate derivative $H(s) = \frac{1-e^{-\tau s}}{\tau}$, (used in the development of the discrete update laws) don't deviate from the true time derivative $d/dt(\cdot)$, with Laplace transform s. For this example, the maximum frequency ω^* of signals is approximately $\omega^* = \pi$ rad/s from Fig. 2. Similar to [13], each signal in the set of $N_s = 100$ signals, used to evaluate the different DAC algorithms, is generated by randomly selecting Fourier coefficients with a frequency spectrum below ω^*, and then the time-domain signal is obtained using the inverse Fourier transform. Additionally, each signal is then filtered with a low pass filter with a cutoff frequency ω^* to ensure the signals are sufficiently smooth. From the $N_s = 100$ signals, the signal that yields the largest error ε is chosen as the reference position signal for each agent.

1) Standard DAC and Accelerated DAC (A-DAC): The update gains $(\gamma, \beta_1, \beta_2 = 0)$ in the standard and accelerated DAC (A-DAC), i.e., Eq. (10), are chosen (using a numerical grid search) to minimize the tracking error metric in Eq. (32) over the set of $N_s = 100$ signals. The range of the selected gains was chosen to satisfy the lower and upper bounds for



(a) Magnitude (b) Phase

Fig. 2: Bode plots of exact and delay-based derivative.



(a) Standard DAC tracking (b) Accelerated DAC tracking

(c) DSR-DAC tracking (d) M-DAC tracking

Fig. 3: Tracking of the random reference signal with the largest error for all methods using the example network in Fig. 1. (a) shows the tracking for the standard DAC method, (b) for the accelerated DAC method, (c) for the DSR-DAC method, and (d) for the momentum DAC method. The red curve denotes the true average of the individual agent's signals, while the blue curves are each agent's estimate of the average signal.

stability, i.e., Eq. (13) for the accelerated DAC and Eq. (14) for the standard DAC. For the standard DAC, with $\beta_1 = \beta_2 = 0$, the optimal update gain was $\gamma = 0.47$ with a tracking error of $\varepsilon = 5.23$, as tabulated in Table I. Fig. 3a shows the tracking of the standard DAC on the random signal with the largest error out of the signal set. It can be observed that each agent's tracking error is significant throughout the entire signal.

For A-DAC, the optimal update gains for the smallest tracking error were $(\gamma, \beta_1, \beta_2 = 0) = (0.93, 0.23, 0)$ with a tracking error of $\varepsilon = 2.87$. Compared to the standard DAC, the A-DAC reduces the tracking error by 45%, as shown in Fig. 3b. Thus, including the outdated feedback term results in more cohesive tracking than the standard DAC method.

2) DSR-DAC: The update gains $(\gamma, \beta_1, \beta_2)$ in the proposed DSR-DAC method, i.e., Eq. (23), are chosen to minimize the tracking error, similar to the standard and accelerated DAC methods. From the numerical grid search, the optimal gains were $(\gamma, \beta_1, \beta_2) = (0.72, -0.11, 0.99)$ with a tracking error of $\varepsilon = 0.47$. The parameter β_2, which is close to $\beta_2 \approx 1$, matches the DSR-DAC for accurate cohesive tracking (as indicated in Section III-D), which was derived from the ideal centralized algorithm. With $\beta_2 = 1$, the tracking error of $\varepsilon = 0.47$ is the same as the case when $\beta_2 = 0.99$, as tabulated in Table I. Compared to the standard DAC and A-DAC, the tracking errors are significantly reduced using the DSR-DAC method by 91% and 84%, respectively, with the tracking of the DSR-DAC shown in Fig. 3c.

3) Momentum DAC (M-DAC): For the momentum DAC (M-DAC), the parameters for the smallest tracking error were $(\gamma, \beta_1 = 0, \beta_2) = (0.9, 0, 0.84)$. This choice of parameters corresponds to a tracking error value of $\varepsilon = 0.53$. This shows that the M-DAC (which is a special case of the generalized DSR-DAC in Eq. (24)) gives comparable performance to (only 12% larger error than) the DSR-DAC without the outdated feedback, as shown in Fig. 3d. However, the choice of parameters for minimal tracking error differs from those for fastest convergence, as will be discussed in the following section.

B. Cohesive tracking versus faster convergence

A faster convergence rate with the DAC algorithm doesn't necessarily imply cohesive tracking of the average value (and vice versa). In the next simulations, the parameters for each DAC algorithm are chosen to have the fastest convergence, i.e., minimizing the convergence rate ρ so that all the poles

TABLE I: Optimizing tracking error

Method	Parameters $(\gamma, \beta_1, \beta_2)$	Tracking error ε on largest signal error	Convergence rate ρ
DAC	$(0.47, 0, 0)$	5.23	0.96
A-DAC	$(0.93, 0.23, 0)$	2.87	0.98
M-DAC	$(0.9, 0, 0.84)$	0.53	0.94
DSR-DAC	(0.72, -0.11, 1)	0.47	0.97

are farthest away from the unit circle. For the standard DAC method, the parameter γ was selected optimally for the fastest convergence, as in Eq. (16), i.e.,

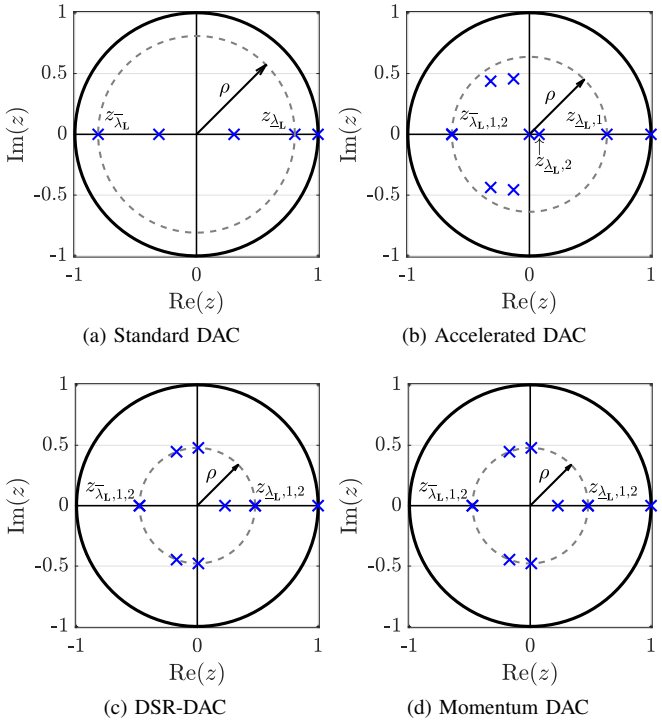
$$\gamma = \frac{2}{\overline{\lambda}_{\mathbf{L}} + \underline{\lambda}_{\mathbf{L}}} = \frac{2}{4.17 + 0.52} = 0.43, \tag{33}$$

where the largest and second smallest Laplacian eigenvalues for the example network in Fig. 1 are $\overline{\lambda}_{\mathbf{L}} = 4.17$ and $\underline{\lambda}_{\mathbf{L}} = 0.52$, respectively. All the poles (except the one at $z_{\lambda_{\mathbf{L}}} = 1$) are inside a circle of radius $\rho = 0.78$, as shown in the root locus in Fig. 4. The accelerated DAC, i.e., the standard method with outdated feedback, achieves faster convergence than the standard method with a convergence rate $\rho = 0.64$. However, this does not correspond to the fastest convergent DAC method, as the parameters for optimal convergence according to the generalized DSR-DAC update law indicate that only momentum is needed from

TABLE II: Optimizing convergence rate ρ

Method	Parameters $(\gamma, \beta_1, \beta_2)$	Tracking error ε on largest signal error	Convergence rate ρ
DAC	$(0.43, 0, 0)$	5.69	0.78
A-DAC	$(0.64, 0.1, 0)$	4	0.64
M-DAC	$(0.52, 0, 0.23)$	1.53	0.48
DSR-DAC	$(0.52, 0, 0.23)$	1.53	0.48

Eq. (29). By only adding momentum to the standard DAC, i.e., $\beta_1 = 0$ and the optimal parameters (γ, β_2) using Eq. (4), i.e., $\gamma = \frac{4}{\left(\sqrt{\lambda_{\mathbf{L}}}+\sqrt{\lambda_{\mathbf{L}}}\right)^2} = \frac{4}{\left(\sqrt{4.17}+\sqrt{0.52}\right)^2} \approx 0.52$, and $\beta_2 = \frac{\left(\sqrt{\lambda_{\mathbf{L}}}-\sqrt{\lambda_{\mathbf{L}}}\right)^2}{\left(\sqrt{\lambda_{\mathbf{L}}}+\sqrt{\lambda_{\mathbf{L}}}\right)^2} = \frac{\left(\sqrt{4.17}-\sqrt{0.52}\right)^2}{\left(\sqrt{4.17}+\sqrt{0.52}\right)^2} \approx 0.23$, the momentum DAC achieves a faster convergence rate, with all the poles (except the one at $z_{\lambda_{\mathbf{L}}} = 1$) inside the circle of radius $\rho = 0.47$, as shown in the root locus in Fig. 4. This choice of parameters is in agreement with the numerical grid search for finding the parameters $(\gamma, \beta_1, \beta_2)$ in the generalized DSR-DAC method. For each case, the parameters that result in the fastest convergence have a convergence rate ρ that is smaller than the convergence rate of the parameters that minimize the tracking error. The choice of parameters for the fastest convergence, however, results in substantially larger tracking errors for each method with an error of $\varepsilon = 5.69$ (9% increase), $\varepsilon = 4$ (39% increase), $\varepsilon = 1.53$ (189% increase), and $\varepsilon = 1.53$ (225% increase) for the standard DAC, A-DAC, M-DAC, and DSR-DAC, respectively. Thus, fast convergence does not necessarily imply cohesive tracking, and both the momentum term and the outdated feedback terms are needed, as in the DSR-DAC, to achieve cohesive tracking.



(a) Standard DAC (b) Accelerated DAC

(c) DSR-DAC (d) Momentum DAC

Fig. 4: Optimizing for fastest convergence: Pole locations, denoted by 'x', and the convergence rate ρ denoted by the gray dashed contour corresponding to the (a) standard DAC (b) momentum DAC, (c) accelerated DAC, and (d) DSR-DAC with the parameters in Table II.

V. Conclusion

This work introduces a novel cohesive-tracking dynamic average consensus (DAC) algorithm using a delayed self reinforcement (DSR) approach. By using local current and past information from the network and reference signals, the proposed DSR-DAC method approximates the ideal centralized DAC, resulting in improved cohesive tracking of the average signal for rapidly changing reference signals. In addition, it was demonstrated that improving the convergence rate in the DAC algorithms does not enhance the tracking performance of rapidly changing trajectories (and vice versa). Numerical simulation results validate the effectiveness of the proposed DSR-DAC approach, showing substantial enhancements in cohesive tracking compared to the standard and accelerated DAC algorithms.

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