2024 Tenth Indian Control Conference (ICC)
9 - 11 December, 2024
IISER Bhopal, India

Controllability of Fractional Delay Partial Random Differential Equations in Banach space

Durga Nagarajan[1*] and Nageshwari S[1]

Abstract— **In this paper, the controllability property for Caputo fractional delay Partial Random Differential Equations (PRDE) in Banach space are investigated. By utilizing the principles from fractional calculus, measures of non-compactness, and fixed point theorems within a stochastic framework, the existence of random mild solutions are established. The Mönch fixed point theorem is then employed to examine the controllability of the proposed system. An example is given to validate the theoretical results.**

Keywords: Random Differential Equations, Fixed Point Theorem, Caputo Derivative, Fractional Delay Differential Equations, Controllability.

I. INTRODUCTION

Fractional calculus deals with derivatives and integrals of non-integer order [1]. Recently, it has been recognized for its significant role across a variety of areas in both practical and theoretical aspects, including heat conduction, viscoelastic systems, finance, robotics, electromagnetic waves, and colored noise [2]. Fractional Differential Systems(FDSs) have emerged as a prominent area of research due to it's wide range of applications across numerous fields in applied sciences and engineering [3]. There are several types of fractional derivatives among which Caputo derivative is particularly noted for its ability to accommodate initial conditions in a way that is consistent with classical differential equations, which makes it especially useful for practical applications in science and engineering [4]. Significant advancements in the analysis of integral equations for ordinary and partial FDSs have been made recently [5]. In Partial neutral integro-differential equations of fractional order were examined by the authors in reference [6] to determine the necessary criteria for the existence of solutions, employing appropriate fixed point theorems.

The task of determining a suitable control function to steer the suggested dynamical model towards a designated end state is known as the controllability problem [7]. With the use of ideas from fixed-point theory, semigroups of operators, fractional calculus, and Measures of Non-Compactness (MNC), in [8], the authors examined the controllability results for Hilfer fractional neutral stochastic differential equations with nonlocal conditions and infinite delay.

This work is supported by Science and Engineering Research Board (SERB), DST, Govt. of India, SPG Project File No. SPG/2021/002891 dated 30.08.2022

[1]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology-Chennai Campus, Chennai - 600 127, Tamil Nadu, India. Email: durga.nagarajan@vit.ac.in.*Corresponding author. nageshwari.s2023@vitstudent.ac.in.

The vitality of a dynamic system in engineering and natural sciences depends on the accuracy of our knowledge of its characteristics. If the knowledge about a dynamic system is precise, a deterministic dynamical system will arise. However, most data for modeling dynamic systems is often incorrect, imprecise, or unclear, leading to uncertainty in determining system parameters. When we have probabilistic knowledge of these parameters, the typical approach in mathematical modeling is to use stochastic or random differential equations [9]. These systems are widely used and studied in various applications. For example, PRDEs are used to model and control systems that are affected by both spatial and temporal variations, along with random disturbances. In real life robotics, for instance, if the robot arm is reaching for an object, it might encounter more resistance at certain points, or the motors could behave differently depending on the angle. To handle this complexity, PRDEs are used to model the uncertainty in the system. By incorporating this randomness into the model, PRDEs help to optimize the robot's path, allowing it to adjust its movements in real-time. This enables the robot to avoid obstacles and reach its destination reliably, even when the environment is uncertain or subject to random changes [10]. In [11], two classes of second-order functional differential equations with delay and random effects are shown to be controllable by the authors, and also investigated the possibility of mild random solutions inside a stochastic domain using a random fixed point theorem.

Within the context of partial fractional random differential equations with infinite delay, the authors in [12], investigated the existence of outcomes for the Darboux problem along with the random fixed point theorem and MNC in a stochastic domain. However, based on the literature study, there is no studies has been done regarding the controllability of fractional delay PRDE in Banach space. To resolve this gap, in this study, we discuss the controllability of the proposed model. The main contributions are highlighted as follows:

- This paper explores a class of fractional delay PRDE in Banach space.
- The controllability findings for the suggested system are examined using the Mönch fixed point theorem, which are addressed in this paper for the first time in the literature.
- The main discoveries are simplified and made easier by using the random fixed point theorem, Kuratowskii MNC, and fractional calculus features within the stochastic framework.

Now, consider the fractional delay partial random differential equation:

$$({}_0^C D^{\xi} x)(\varsigma,\nu,\tau) = x(\varsigma,\nu,\tau) + \psi(\varsigma,\nu,x_{(\varsigma,\nu)},\tau) + Bu(\varsigma,\nu,\tau),\ \text{if}(\varsigma,\nu) \in \Theta := [0,\theta_1]\times[0,\theta_2],\ \tau \in \chi \quad (1)$$

$$x(\varsigma,\nu,\tau) = \varpi(\varsigma,\nu,\tau),\ \text{if } (\varsigma,\nu) \in \breve{\Theta} := (-\infty,\theta_1]\times(-\infty,\theta_2]\setminus\Theta, \tau\in\chi, \quad (2)$$

$$x(\varsigma,0,\tau) = \varpi_1(\varsigma,\tau),\ \varsigma\in[0,\theta_1],\ x(0,\nu,\tau) = \varpi_2(\nu,\tau),\ \nu\in[0,\theta_2], \quad (3)$$

where $(\chi,\mathscr{F},P)$ represent a complete probability space, with $\theta_1,\theta_2>0$. The symbol ${}^C D_0^{\xi}$ denotes the standard Caputo's fractional derivative of order $\xi=(\xi_1,\xi_2)\in(0,1]\times(0,1]$. Consider $(\mathfrak{Y},\|\cdot\|)$ as a Banach space, and $(\chi,\mathbb{A})$ as a measurable space. The function $\psi:\Theta\times\mathfrak{K}\times\chi\to\mathfrak{Y}$ is specified, where $\mathfrak{K}$ is a phase space. The control function $u(\cdot,\cdot,\tau)$ is defined in $L^2(\Theta,\Lambda)$, which is a Banach space of admissible control functions with Λ. Let $\varpi:\breve{\Theta}\times\chi\to\mathfrak{Y}$ be a given continuous function, with $\varpi_1:[0,\theta_1]\times\chi\to\mathfrak{Y}$, $\varpi_2:[0,\theta_2]\times\chi\to\mathfrak{Y}$ are absolutely continuous functions with $\varpi_1(\varsigma,\tau)=\varpi(\varsigma,0,\tau),\varpi_2(\nu,\tau)=\varpi(0,\nu,\tau)$ for each $\varsigma\in[0,\theta_1],\nu\in[0,\theta_2],\tau\in\chi$. Let u be a bounded linear operator from Λ into $\mathfrak{Y}$. Here, $x_{(\varsigma,\nu)}(\cdot,\cdot,\tau)$ denotes the history of the state from $-\infty$ to the present time (ς,ν). Let $x_{(\varsigma,\nu)}\in\mathfrak{K}$ which gives

$$x_{(\varsigma,\nu)}(\rho,k,\tau) = x(\varsigma+\rho,\nu+k,\tau);(\rho,k)\in(-\infty,0]\times(-\infty,0].$$

This paper has three sections, which is structured as follows: Section 2 demonstrates sufficient lemmas, notations, and some concepts. Section 3 presents the main results for the controllability of the proposed system.

II. Preliminaries

We gather various notions, lemmas, and definitions that we will utilize in the main results.

Let $AC(\Theta)$ be the space that contains the functions which are absolutely continuous mapping from Θ to $\mathfrak{Y}$. Consider the space $L^1(\Theta)$ that consists of Bochner-integrable functions $x:\Theta\to\mathfrak{Y}$ with the norm

$$\|x\|_{L^1} = \int_0^{\theta_1}\int_0^{\theta_2}\|x(\varsigma,\nu)\|_{\mathfrak{Y}}\,d\nu\,d\varsigma.$$

Consider the Banach space $L^{\infty}(\Theta)$ of functions $x:\Theta\to\mathbb{R}$ with bounded norm

$$\|x\|_{L^{\infty}} = \sup_{(\varsigma,\nu)\in\Theta}|x(\varsigma,\nu)|.$$

We will employ an axiomatic definition of phase space $\mathfrak{K}$ and adhere to the terminology established in [13]. For additional examples, please read the book [14] and its references. Let $(\mathfrak{K},\|\cdot\|_{\mathfrak{K}})$ be seminormed linear space of functions mapping from $(-\infty,0]\times(-\infty,0]\to\mathbb{R}^n$, and with the axioms:

S1 : If $x:(-\infty,\theta_1]\times(-\infty,\theta_2]\to\mathbb{R}^n,\theta_1,\theta_2>0$ continuous on Θ and $x_{(\varsigma,\nu)}\in\mathfrak{K}$, then $\forall\ (\varsigma,\nu)\in\Theta$, the following assumption are met:

a) $x_{(\varsigma,\nu)}\in\mathfrak{K}$;

b) $\exists$ two functions $\gamma(\cdot),\sigma(\cdot):\mathbb{R}_+\to\mathbb{R}_+$ independent of x with γ, continuous and bounded $\ni$

$$\|x_{(\varsigma,\nu)}\|_{\mathfrak{K}} \le \gamma(\varsigma,\nu)\sup\{|x(u,v)|:0\le(u,v)\le(\varsigma,\nu)\}+\sigma(\varsigma,\nu)\|x_{(\varsigma,\nu)}\|_{\mathfrak{K}}.$$

S2 : For the function x in (**S1**), $x_{(\varsigma,\nu)}$ is a $\mathfrak{K}$-valued continuous function on Θ.

S3 : The space $\mathfrak{K}$ is complete.

Consider the space

$$\mathfrak{A} = \{x:(-\infty,\theta_1]\times(-\infty,\theta_2]\to\mathfrak{Y} : x|_{(-\infty,0]\times(-\infty,0]}\in\mathfrak{K} \text{ and } x|_{\Theta} \text{ is continuous }\}.$$

Consider the seminorm $\|x\|_{\mathfrak{A}}$ in $\mathfrak{A}$ given by

$$\|x\|_{\mathfrak{A}} = \|\varpi\|_{\mathfrak{K}} + \|x\|_{L^{\infty}}.$$

Let $\mathfrak{K}_{\mathfrak{Y}}$ be the $\sigma-$algebra of Borel subsets of $\mathfrak{Y}$. If for any $\Lambda\in\mathfrak{K}_{\mathfrak{Y}}$, then the map $\mathscr{P}:\chi\to\mathfrak{Y}$ is measurable. We have

$$\mathscr{P}^{-1}(\Lambda) = \{\tau\in\chi:\mathscr{P}(\tau)\in\Lambda\}\subset\mathbb{A}.$$

Definition 2.1: A mapping $\mathscr{A}:\chi\times\mathfrak{Y}\to\mathfrak{Y}$ is jointly-measurable if for any $\Lambda\in\mathfrak{K}_{\mathfrak{Y}}$, we have

$$\mathscr{A}^{-1}(\Lambda) = \{(\tau,\mathscr{P})\in\chi\times\mathfrak{Y}:\mathscr{A}(\tau,\mathscr{P})\in\Lambda\}\subset\mathbb{A}\times\mathfrak{K}_{\mathfrak{Y}}.$$

Here, $\mathbb{A}$ and $\mathfrak{K}_{\mathfrak{Y}}$ is defined in χ and $\mathfrak{Y}$ respectively. Where $\mathbb{A}\times\mathfrak{K}_{\mathfrak{Y}}$ is the direct product of the σ-algebra $\mathbb{A}$ and $\mathfrak{K}_{\mathfrak{Y}}$.

Lemma 2.2: Consider a mapping be $\mathscr{A}:\chi\times\mathfrak{Y}\to\mathfrak{Y}$. The map $(\tau,\mathscr{P})\to\mathscr{A}(\tau,\mathscr{P})$ is jointly- measurable if $\mathscr{A}(\cdot,\mathscr{P})$ is measurable $\forall\ \mathscr{P}\in\mathfrak{Y}$, and $\mathscr{A}(\tau,\cdot)$ is continuous $\forall\ \tau\in\chi$.

Definition 2.3: A function $\psi:\Theta\times\mathfrak{Y}\times\chi\to\mathfrak{Y}$ is said to be random carathéodary if the hypotheses that follows:

- The map $(\varsigma,\nu,\tau)\to\psi(\varsigma,\nu,x_{(\varsigma,\nu)},\tau)$ is jointly-measurable $\forall\ x_{(\varsigma,\nu)}\in\mathfrak{Y}$.
- $x\to\psi(\varsigma,\nu,x,\tau)$ is continuous $\forall\ (\varsigma,\nu)\in\Theta$ and $\tau\in\chi$.

If $\mathscr{A}(\tau,x)$ is measurable in $\tau\ \forall\ x\in\mathfrak{Y}$ and it is given as $\mathscr{A}(\tau)x=\mathscr{A}(\tau,x)$. Then the map $\mathscr{A}:\chi\times\mathfrak{Y}$ is a random operator and also say that $\mathscr{A}(\tau)$ is a random operator on $\mathfrak{Y}$.

Let $\varepsilon=(0,0),\xi_1,\xi_2>0$ and $\xi=(\xi_1,\xi_2)$. For $\psi\in L^1(\Theta)$, the left-sided mixed Riemann-Liouville integral of order ξ is given by:

$$(I_{\varepsilon}^{\xi}\psi)(\varsigma,\nu) = \frac{1}{\Gamma(\xi_1)\Gamma(\xi_2)}\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\times\psi(\rho,k)dkd\rho.$$

In particular,

$$(I_{\varepsilon}^{0}x)(\varsigma,\nu) = x(\varsigma,\nu),$$

$$(I_{\varepsilon}^{w}x)(\varsigma,\nu) = \int_0^{\varsigma}\int_0^{\nu}x(\rho,k)dkd\rho;\ \forall\ (\varsigma,\nu)\in\Theta,$$

where $w=(1,1)$ and $(1-\xi)$ means $(1-\xi_1,1-\xi_2)\in[0,1)\times[0,1)$.

Definition 2.4: Let $\xi\in(0,1]\times(0,1]$ and $x\in AC(\Theta)$. The Caputo fractional derivative of order ξ of x is given by:

$${}^C D_{\varepsilon}^{\xi}x(\varsigma,\nu) = (I_{\varepsilon}^{1-\xi}D_{\varsigma\nu}^{2}x)(\varsigma,\nu).$$

The case $w=(1,1)$ is included and one can get

$$({}^{C}D^{w}_{\varepsilon}x)(\varsigma,\nu)=(D^{2}_{\varsigma\nu}x)(\varsigma,\nu);\ \forall\ (\varsigma,\nu)\in\Theta.$$

Lemma 2.5: refer [15], [16] Let $\varphi\in L^{1}(\Theta)$. The linear problem

$$\begin{aligned}{}^{C}D^{\xi}_{\varepsilon}x(\varsigma,\nu)&=\varphi(\varsigma,\nu);\ \forall\ (\varsigma,\nu)\in\Theta=[0,\theta_1]\times[0,\theta_2],\\ x(\varsigma,0)&=\varpi(\varsigma);\varsigma\in[0,\theta_1],\\ x(0,\nu)&=\varpi(\nu);\nu\in[0,\theta_2],\\ \varpi_1(0)&=\varpi_2(0)\end{aligned}$$

has the following unique solution:
$x(\varsigma,\nu)=P(\varsigma,\nu)+I^{\xi}_{\varepsilon}\varphi(\varsigma,\nu);\ \forall\ (\varsigma,\nu)\in\Theta$, where $P(\varsigma,\nu)=\varpi_1(\varsigma)+\varpi_2(\nu)-\varpi_1(0)$.

Lemma 2.6: [10] Let $\mathscr{G}:\chi\to 2^{\mathfrak{Y}}$ be measurable with $\mathscr{G}(\tau)$ solid, closed, and convex $(i.e.,int\mathscr{G}(\tau)\neq\emptyset)\ \forall\ \tau\in\chi$. We suppose that $\exists$ measurable $q_0:\chi\to\mathfrak{Y}$ with $q_0\in int\mathscr{G}(\tau)\ \forall\ \tau\in\chi$. Consider the continuous random operator G with stochastic domain $\mathscr{G}(t)$ such that $\forall\ \tau\in\chi,\{\acute{q}\in\mathscr{G}(\tau):G(\tau)\acute{q}=\acute{q}\}\neq\emptyset$. Then the operator G has a stochastic fixed point.

Definition 2.7: [17] Consider the Banach space Π. Let $\chi_{(\Pi)}$ be the bounded subsets of $\mathfrak{Y}$. The Kuratowskii MNC is the map μ from $\chi_{(\Pi)}\to[0,\infty)$ given by

$$\begin{aligned}\mu(\Upsilon)=&\inf\{\varepsilon>0:\Upsilon\subseteq\bigcup_{i=1}^{n}\Upsilon_i\text{ and }\operatorname{diam}(\Upsilon_i)\leq\varepsilon\};\\ &\text{here }\Upsilon\in\chi_{(\Pi)},\end{aligned}$$

and verifies the properties:

1) $\mu(\Upsilon)=0$ iff $\acute{\Upsilon}$(Υ is relatively compact);
2) $\mu(\varepsilon\Upsilon)=|\varepsilon|\mu(\Upsilon);\varepsilon\in\mathbb{R}$;
3) $\mu(\acute{\Upsilon}+\Upsilon)\leq\mu(\acute{\Upsilon})+\mu(\Upsilon)$;
4) $\mu(\Upsilon)=\mu(\acute{\Upsilon})$;
5) $\acute{\Upsilon}\subset\Upsilon$ implies $\mu(\acute{\Upsilon})\leq\mu(\Upsilon)$;
6) $\mu(conv\Upsilon)=\mu(B)$.

Lemma 2.8: [18] If $\acute{g}\subset L^{1}(\Theta)$, then $\mu(\acute{g}(\varsigma,\nu))$ is measurable and for each $(\varsigma,\nu)\in\Theta$,

$$\begin{aligned}\mu\left(\left\{\int_0^{\varsigma}\int_0^{\nu}g(\rho,k)dkd\rho:g\in\acute{g}\right\}\right)&\\ \leq\int_0^{\varsigma}\int_0^{\nu}\mu\left(\left\{g(\rho,k)dkd\rho:g\in\acute{g}\right\}\right),&\end{aligned}$$

where μ represents Kuratowskii MNC on the space $\mathfrak{Y}$.
Let $0<\xi_1,\xi_2\leq 1,p(\varsigma,\nu,\tau)=\varpi_1(\varsigma,\tau)+\varpi_2(\nu,\tau)-\varpi_1(0,\tau)$. According to Lemma 2.5, we now state that a function $x\in\chi\times\mathfrak{A}$ is a solution of $(1)-(3)$ if x fulfills (2) for $(\varsigma,\nu)\in\breve{\Theta},\tau\in\chi$ and x is a solution of the proposed system

$$\begin{aligned}x(\varsigma,\nu,\tau)=P(\varsigma,\nu,\tau)+\int_0^{\varsigma}\int_0^{\nu}\frac{(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}}{\Gamma(\xi_1)\Gamma(\xi_2)}&\\ \psi(\rho,k,x_{(\rho,k)},\tau)dkd\rho+\int_0^{\varsigma}\int_0^{\nu}\frac{(\nu-\rho)^{\xi_1-1}(\varsigma-k)^{\xi_2-1}}{\Gamma(\xi_1)\Gamma(\xi_2)}&\\ \times Bu(\varsigma,\nu,\tau)dkd\rho.&\end{aligned}$$

Definition 2.9: The problem $(1)-(3)$ is controllable on the interval $(-\infty,\theta_1]\times(-\infty,\theta_2]$, if for every final state $x_1(\tau)$, there exists a control $u(\varsigma,\nu,\tau)$ is defined in $L^{2}(\Theta,\Lambda)$, such that the solution x of $(1)-(3)$ verifies (2).

III. Main results

The hypotheses that are needed to support the article's primary results are as follows:

H1 : The function $\tau\to\varpi_1(\varsigma,0,\tau)$ and $\tau\to\varpi_2(0,\nu,\tau)$ are measurable and bounded for $(\varsigma,\nu)\in\Theta$.

H2 : The functions ϖ is measurable for $(\varsigma,\nu)\in\breve{\Theta}$.

H3 : The function ψ is random carathéodary on $\Theta\times\mathfrak{K}\times\chi$.

H4 : The linear operator $K:L^{2}(\Theta,\Lambda)\to\mathfrak{Y}$ is given by:

$$Ku=\int_0^{\varsigma}\int_0^{\nu}\frac{(\nu-\rho)^{\xi_1-1}(\varsigma-k)^{\xi_2-1}}{\Gamma(\xi_1)\Gamma(\xi_2)}Bu(\varsigma,\nu,\tau)dkd\rho$$

has an inverse operator K^{-1} in $L^{2}(\Theta,\Lambda)/kerK$ and $\exists$ a constant $\eta>0$ such that $\|BK^{-1}\|\leq\eta$.

H5 : $\exists$ a function $\sigma_1,\sigma_2:\Theta\times\chi\to[0,\infty)$ with $\sigma_i(\cdot,\tau)\in L^{\infty}(\Theta,[0,\infty));\ i=1,2$, such that for each $\tau\in\chi$,

$$\|\psi(\varsigma,\nu,x_{(\varsigma,\nu)},\tau)\|_{\mathfrak{Y}}\leq\sigma_1(\varsigma,\nu,\tau)+\sigma_2(\varsigma,\nu,\tau)\|x_{(\varsigma,\nu)}\|_{\mathfrak{K}}$$

$\forall\ x_{(\varsigma,\nu)}\in\mathfrak{K}$ and a.e. $(\varsigma,\nu)\in\Theta$.

H6 : For any bounded $\Lambda\subset\mathfrak{Y}$,
$\mu(\chi(\varsigma,\nu,\Lambda,\tau))\leq\sigma_2(\varsigma,\nu,\tau)\mu(\Lambda),\ \forall\ (\varsigma,\nu)\in\Theta$, and $\sigma^{*}_{j}(\tau)=\sup ess_{(\varsigma,\nu)\in\Theta}\sigma_j(\varsigma,\nu,\tau);j=1,2$.

H7 : $\exists$ a random function $Q:\chi\to\mathbb{R}^{+}\setminus\{0\}$ and constants $a,b>0$, such that

$$\begin{aligned}&w_0+\frac{\varsigma^{\xi_1}\nu^{\xi_2}(\sigma^{*}_1(\tau)+\sigma^{*}_2(\tau)\rho^{*}(\tau))}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\\ &+\frac{\eta\varsigma^{\xi_1}\nu^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\times\left[\|x_1(\tau)\|\right.\\ &\left.+w_0+\frac{a^{\xi_1}b^{\xi_2}(\sigma^{*}_1(\tau)+\sigma^{*}_2(\tau)\rho^{*}(\tau))}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\right]\leq Q(\tau).\end{aligned}$$

Theorem 3.1: Suppose that $H1-H7$ fulfilled, then the system $(1)-(3)$ is controllable on Θ, provided

$$\frac{\varsigma^{\xi_1}\nu^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\sigma_2(\rho,k,\tau)(1+\eta\ell_0)<1.$$

Proof: By using the assumption $H4$, we define the control $u(\varsigma,\nu,\tau)$ by

$$\begin{aligned}u(\varsigma,\nu,\tau)=&K^{-1}\left[x_1(\tau)-P(a,b,\tau)+\int_0^{a}\int_0^{b}\frac{(a-p)^{\xi_1-1}(b-q)^{\xi_2-1}}{\Gamma(\xi_1)\Gamma(\xi_2)}\right.\\ &\left.\times\psi(p,q,x_{(p,q)},\tau)dqdp\right].\end{aligned}$$

We define the operator $\mathscr{F}:\chi\times\mathfrak{A}\to\mathfrak{A}$ by

$$(\mathscr{F}(\tau)x)(\varsigma,\nu)=\begin{cases}\varpi(\varsigma,\nu,\tau),\ \text{if }(\varsigma,\nu)\in\breve{\Theta}\\ P(\varsigma,\nu,\tau)+\int_0^{\varsigma}\int_0^{\nu}\frac{(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}}{\Gamma(\xi_1)\Gamma(\xi_2)}\\ \times\psi(\rho,k,x_{(\rho,k)},\tau)dkd\rho\\ +\int_0^{\varsigma}\int_0^{\nu}\frac{(\nu-\rho)^{\xi_1-1}(\varsigma-k)^{\xi_2-1}}{\Gamma(\xi_1)\Gamma(\xi_2)}BK^{-1}\\ \times\left[x_1(\tau)-P(a,b,\tau)\right.\\ +\int_0^{a}\int_0^{b}\frac{(a-p)^{\xi_1-1}(b-q)^{\xi_2-1}}{\Gamma(\xi_1)\Gamma(\xi_2)}\\ \left.\times\psi(p,q,x_{(p,q)},\tau)dqdp\right]dkd\rho,\\ \text{if }(\varsigma,\nu)\in\Theta.\end{cases}$$

Let $A:\chi\to 2^{\mathfrak{A}}$ is defined by

$$A(\tau)=\{x\in\mathfrak{A}:\|x\|_{\mathfrak{A}}\le Q(\tau)\}.$$

$A(\tau)$ is closed, bounded, solid and convex $\forall\ \tau\in\chi$. Then A is measurable by Lemma 17 in [19] . Let $\tau\in\chi$ be fixed. If $x\in A(\tau)$, and by $(S1)$, then

$$\|x_{(\rho,k)}\|_{\mathfrak{K}}\le\gamma(\rho,k)|x(u,v)|+\sigma(\rho,k)\|\varpi\|_{\mathfrak{K}}. \tag{4}$$

Claim 1. $\mathscr{F}(\tau)$ is a random operator with stochastic domain on A.

Let $\tau\in\chi$ be fixed, then $\forall\ x\in A(\tau)$, and by $H4,H5,H7$, and (4), we have

$$\begin{aligned}
&\|(\mathscr{F}(\tau)x)(\varsigma,\nu)\|\\
\le&\|P(\varsigma,\nu,\tau)\|+\int_0^{\varsigma}\int_0^{\nu}\frac{(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}}{\Gamma(\xi_1)\Gamma(\xi_2)}\\
&\times\|\psi(\rho,k,x_{(\rho,k)},\tau)\|dkd\rho\\
&+\int_0^{\varsigma}\int_0^{\nu}\frac{(\nu-\rho)^{\xi_1-1}(\varsigma-k)^{\xi_2-1}}{\Gamma(\xi_1)\Gamma(\xi_2)}\\
&\times\|BK^{-1}\|\Bigg[\|x_1(\tau)\|+\|P(a,b,\tau)\|\\
&+\int_0^{a}\int_0^{b}\frac{(a-p)^{\xi_1-1}(b-q)^{\xi_2-1}}{\Gamma(\xi_1)\Gamma(\xi_2)}\\
&\times\|\psi(p,q,x_{(p,q)},\tau)\|dqdp\Bigg]dkd\rho\\
\le&w_0+\frac{\varsigma^{\xi_1}\nu^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\\
&\times\left[\sigma_1(\rho,k,\tau)+\sigma_2(\rho,k,\tau)\|x_{(\rho,k)}\|_{\mathfrak{K}}\right]\\
&+\frac{\eta\varsigma^{\xi_1}\nu^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\times\Bigg[\|x_1(\tau)\|+w_0\\
&+\frac{a^{\xi_1}b^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\\
&\times\left[\sigma_1(p,q,\tau)+\sigma_2(p,q,\tau)\|x_{(p,q)}\|_{\mathfrak{K}}\right]\Bigg]\\
\le&w_0+\frac{\varsigma^{\xi_1}\nu^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\Bigg[\sigma_1(\rho,k,\tau)+\sigma_2(\rho,k,\tau)\\
&\times\ [\gamma(\rho,k)|x(u,v)|+\sigma(u,v)\|\varpi\|_{\mathfrak{K}}]\Bigg]+\frac{\eta\varsigma^{\xi_1}\nu^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\\
&\times\Bigg[\|x_1(\tau)\|+w_0+\frac{a^{\xi_1}b^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\big[\sigma_1(p,q,\tau)\\
&+\sigma_2(p,q,\tau)\ \times[\gamma(p,q)|x(u,v)|+\sigma(u,v)\|\varpi\|_{\mathfrak{K}}]\big]\Bigg]\\
\le&w_0+\frac{\varsigma^{\xi_1}\nu^{\xi_2}(\sigma_1^*(\tau)+\sigma_2^*(\tau)\rho^*(\tau))}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\\
&+\frac{\eta\varsigma^{\xi_1}\nu^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\\
&\times\left[\|x_1(\tau)\|+w_0+\frac{a^{\xi_1}b^{\xi_2}(\sigma_1^*(\tau)+\sigma_2^*(\tau)\rho^*(\tau))}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\right]\\
\le&Q(\tau).
\end{aligned}$$

Thus, $\mathscr{F}$ is a random operator with stochastic domain A and $\mathscr{F}(\tau):A(\tau)\to A(\tau)$ for each $\tau\in\chi$. Moreover, $\mathscr{F}(\tau)$ maps to itself.

Claim 2. $\mathscr{F}(\tau)$ is continuous.

Consider a sequence $\{x^n\}$ such that $x^n\to x$ in $A(\tau)$. Hence, $\forall\ (\varsigma,\nu)\in\Theta$ and $\tau\in\chi$, by using $H3$, we get

$$\begin{aligned}
&\|(\mathscr{F}(\tau)x^n)(\varsigma,\nu)-(\mathscr{F}(\tau)x)(\varsigma,\nu)\|_{\mathfrak{Y}}\\
\le&\frac{1}{\Gamma(\xi_1)\Gamma(\xi_2)}\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\\
&\|\psi(\rho,k,x^n_{(\rho,k)}-\psi(\rho,k,x_{(\rho,k)}\|dkd\rho\\
&+\frac{\eta}{\Gamma(\xi_1)\Gamma(\xi_2)}\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\\
&\times\frac{a^{\xi_1}b^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\\
&\times\|\psi(p,q,x^n_{(p,q)},\tau)-\psi(p,q,x_{(p,q)}\|dkd\rho\\
\le&\left(1+\frac{\eta a^{\xi_1}b^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\right)\frac{1}{\Gamma(\xi_1)\Gamma(\xi_2)}\\
&\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\\
&\times\|\psi(\rho,k,x^n_{(\rho,k)}-\psi(\rho,k,x_{(\rho,k)}\|dkd\rho.
\end{aligned}$$

Thus, $\|\mathscr{F}(\tau)x^n-\mathscr{F}(\tau)x\|_{\infty}\to 0$ as $n\to\infty$. Thus, we can conclude that $\mathscr{F}(\tau):A(\tau)\to A(\tau)$ is a continuous random operator with stochastic domain A.

Now, we demonstrate that $\forall\ \ \tau\in\chi,\{x\in A(\tau):\mathscr{F}(\tau)x=x\}\neq\emptyset$ by employing Mönch fixed point theorem.

Claim 3. $\mathscr{F}$ maps bounded sets into equicontinuous sets in $A(\tau)$.

Let $(x_1,y_1),(x_2,y_2)\in(0,\theta_1]\times(0,\theta_2]$, where $x_1<x_2,y_1<y_2$, then $A(\tau)$ be a bounded set as in claim 2. Let $x\in A(\tau)$, then

$$\begin{aligned}
&\|(\mathscr{F}(\tau)x)(x_2,y_2)-(\mathscr{F}(\tau)x)(x_1,y_1)\|\\
\le&\frac{1}{\Gamma(\xi_1)\Gamma(\xi_2)}\Bigg\|\int_0^{x_1}\int_0^{y_1}\big[(x_2-\rho)^{\xi_1-1}(y_2-k)^{\xi_2-1}\\
&-(x_1-\rho)^{\xi_1-1}(y_1-k)^{\xi_2-1}\big]\\
&\times\big[\psi(\rho,k,x_{(\rho,k)},\tau)+Bu(\varsigma,\nu,\tau)\big]dkd\rho\Bigg\|\\
&+\frac{1}{\Gamma(\xi_1)\Gamma(\xi_2)}\Bigg\|\int_{x_1}^{x_2}\int_{y_1}^{y_2}(x_2-\rho)^{\xi_1-1}(y_2-k)^{\xi_2-1}\\
&\times\big[\psi(\rho,k,x_{(\rho,k)},\tau)+Bu(\varsigma,\nu,\tau)\big]dkd\rho\Bigg\|\\
&+\frac{1}{\Gamma(\xi_1)\Gamma(\xi_2)}\Bigg\|\int_0^{x_1}\int_{y_1}^{y_2}(x_2-\rho)^{\xi_1-1}(y_2-k)^{\xi_2-1}\\
&\times\big[\psi(\rho,k,x_{(\rho,k)},\tau)+Bu(\varsigma,\nu,\tau)\big]dkd\rho\Bigg\|\\
&+\frac{1}{\Gamma(\xi_1)\Gamma(\xi_2)}\Bigg\|\int_{x_1}^{x_2}\int_0^{y_1}(x_2-\rho)^{\xi_1-1}(y_2-k)^{\xi_2-1}\\
&\times\big[\psi(\rho,k,x_{(\rho,k)},\tau)+Bu(\varsigma,\nu,\tau)\big]dkd\rho\Bigg\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\big[y_2^{\xi_2}(x_2-x_1)^{\xi_1}+x_2^{\xi_1}(y_2-y_1)^{\xi_2}\\
&\quad-(x_2-x_1)^{\xi_1}(y_2-y_1)^{\xi_2}+x_1^{\xi_1}y_1^{\xi_2}-x_2^{\xi_1}y_2^{\xi_2}\big]\\
&\quad\times\big[\psi(\rho,k,x_{(\rho,k)},\tau)+Bu(\varsigma,\nu,\tau)\big]\\
&\quad+\frac{1}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}(x_2-x_1)^{\xi_1}(y_2-y_1)^{\xi_2}\\
&\quad\times\big[\psi(\rho,k,x_{(\rho,k)},\tau)+Bu(\varsigma,\nu,\tau)\big]\\
&\quad+\frac{1}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\big[x_2^{\xi_1}-(x_2-x_1)^{\xi_1}\big](y_2-y_1)^{\xi_2}\\
&\quad\times\big[\psi(\rho,k,x_{(\rho,k)},\tau)+Bu(\varsigma,\nu,\tau)\big]\\
&\quad+\frac{1}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}(x_2-x_1)^{\xi_1}y_2^{\xi_2}\\
&\quad\times\big[\psi(\rho,k,x_{(\rho,k)},\tau)+Bu(\varsigma,\nu,\tau)\big]\\
&\leq\frac{1}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\big[\psi(\rho,k,x_{(\rho,k)},\tau)+Bu(\varsigma,\nu,\tau)\big]\\
&\quad\times\Big[2y_2^{\xi_2}(x_2-x_1)^{\xi_1}+2x_2^{\xi_1}(y_2-y_1)^{\xi_2}\\
&\quad+x_1^{\xi_1}y_1^{\xi_2}-x_2^{\xi_1}y_2^{\xi_2}-2(x_2-x_1)^{\xi_1}(y_2-y_1)^{\xi_2}\Big].
\end{aligned}
$$

Therefore, $\|(\mathscr{F}(\tau)x)(x_2,y_2)-(\mathscr{F}(\tau)x)(x_1,y_1)\|\to 0$ as $x_1\to x_2$, and $y_1\to y_2$ by using absolute continuity of Lebesgue dominance theorem. Thus, $\mathscr{F}(A(\tau))$ is equicontinuous on $\mathfrak{A}$.

Claim 4. The Mönch condition is verified here. Let $\phi\subset A(\tau)$ where $\phi(\tau)\subset\overline{conv}\{((\mathfrak{F}\tau)\phi)\cup\{0\}\}$. Clearly, ϕ is bounded and equicontinuous. Thus, the function $(\varsigma,\nu)\to\gamma(\varsigma,\nu)=\mu(\phi\tau)(\varsigma,\nu)$ is continuous on $(-\infty,\theta_1]\times(-\infty,\theta_2]$, and by using Lemma 2.8, and $H6$, we have

$$
\begin{aligned}
&\gamma(\varsigma,\nu)=\mu(\phi(\varsigma,\nu))\leq\mu(((\mathfrak{F}\tau)\phi)(\varsigma,\nu)\cup\{0\})\\
&\leq\mu(((\mathfrak{F}\tau)\phi)(\varsigma,\nu))\\
&\leq\Big(\int_0^{\varsigma}\int_0^{\nu}\frac{(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}}{\Gamma(\xi_1)\Gamma(\xi_2)}\\
&\quad\times\psi(\rho,k,x_{(\rho,k)},\tau)dkd\rho\\
&\quad+\eta\int_0^{\varsigma}\int_0^{\nu}\frac{(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}}{\Gamma(\xi_1)\Gamma(\xi_2)}\\
&\quad\times\left[\frac{a^{\xi_1}b^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\psi(p,q,x_{(p,q)},\tau)\right]dkd\rho\Big)\\
&\leq\frac{1}{\Gamma(\xi_1)\Gamma(\xi_2)}\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\\
&\quad\times\mu\big(\psi(\rho,k,x_{(\rho,k)},\tau)\big)dkd\rho\\
&\quad+\frac{\eta\ell_0}{\Gamma(\xi_1)\Gamma(\xi_2)}\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\\
&\quad\times\mu\big(\psi(p,q,x_{(p,q)},\tau)\big)dkd\rho\\
&\leq\frac{1}{\Gamma(\xi_1)\Gamma(\xi_2)}\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\sigma_2(\rho,k,\tau)\\
&\quad\times\mu\big(x_{(\rho,k)}:x\in\phi\big)dkd\rho\\
&\quad+\frac{\eta\ell_0}{\Gamma(\xi_1)\Gamma(\xi_2)}\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\sigma_2(p,q,\tau)\\
&\quad\times\mu\big(x_{(p,q)}:x\in\phi\big)dkd\rho\\
&\leq\frac{1}{\Gamma(\xi_1)\Gamma(\xi_2)}\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\sigma_2(\rho,k,\tau)\\
&\quad\times\mu(\phi(\rho,k))dkd\rho\\
&\quad+\frac{\eta\ell_0}{\Gamma(\xi_1)\Gamma(\xi_2)}\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\sigma_2(p,q,\tau)\\
&\quad\times\mu(\phi(p,q))dkd\rho\\
&\leq\frac{(1+\eta\ell_0)}{\Gamma(\xi_1)\Gamma(\xi_2)}\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\sigma_2(\rho,k,\tau)\\
&\quad\times\gamma(\rho,k)dkd\rho\\
&\leq\frac{(1+\eta\ell_0)}{\Gamma(\xi_1)\Gamma(\xi_2)}\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\sigma_2(\rho,k,\tau)\\
&\quad\times\sup_{0\leq(\nu_1,\nu_2)\leq(\rho,k)}\gamma(\rho,k)dkd\rho\\
&\leq\frac{(1+\eta\ell_0)}{\Gamma(\xi_1)\Gamma(\xi_2)}\|\gamma\|_{\infty}\int_0^{\varsigma}\int_0^{\nu}(\varsigma-\rho)^{\xi_1-1}(\nu-k)^{\xi_2-1}\\
&\quad\times\sigma_2(\rho,k,\tau)dkd\rho.
\end{aligned}
$$

Thus

$$
\|\gamma\|_{\infty}\leq\frac{\varsigma^{\xi_1}\nu^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\sigma_2(\rho,k,\tau)\|\gamma\|_{\infty}(1+\eta\ell_0).
$$

Then

$$
\|\gamma\|_{\infty}\left(1-\aleph\right)\leq 0,
$$

where, $\frac{\varsigma^{\xi_1}\nu^{\xi_2}}{\Gamma(1+\xi_1)\Gamma(1+\xi_2)}\sigma_2(\rho,k,\tau)(1+\eta\ell_0)$. Thus, $\|\gamma\|_{\infty}=0$, which implies that $\gamma(\varsigma,\nu)=0\ \forall\ (\varsigma,\nu)\in\Theta$. Consequently, $\phi(\varsigma,\nu)$ is relatively compact in $\mathfrak{Y}$. Clearly, ϕ is relatively compact in $A(\tau)$ according to the Ascoli-Arzela theorem. With the help of Mönch fixed point theorem, we can deduce that $\mathscr{F}$ has a fixed point, say $x(\cdot,\cdot,\tau)\in A(\tau)$. Since $\bigcap_{\tau\in\chi}A(\tau)\neq\emptyset$, and there is a measurable selector of $intA$. By Lemma 2.6, $\mathscr{F}$ has a stochastic fixed point which ensures the existence of at least one random solution for the proposed model. Hence, by definition 2.9, the proposed control system $(1)-(3)$ is controllable on $(-\infty,\theta_1]\times(-\infty,\theta_2]$. ■

IV. EXAMPLE

Let $\mathfrak{Y}=\mathbb{R},\chi=(-\infty,0)$ be equipped with the usual $\sigma-$ algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$. Consider the system as follows:

$$
\begin{aligned}
&({}_{\varepsilon}^{C}D^{\xi}x)(\varsigma,\nu,\tau)=x(\varsigma,\nu,\tau)\\
&+\left(\frac{\sin^2|\tau|}{5+\tau^2}+\frac{\|x_{(\varsigma,\nu)}\|}{3+\tau^2+\|x_{(\varsigma,\nu)}\|}\right)\\
&+Bu(\varsigma,\nu,\tau)\ a.a.\ (\varsigma,\nu)\in\Theta=[0,1]\times[0,1],\tau\in\chi,\qquad(5)\\
&x(\varsigma,\nu,\tau)=\varsigma^2\sin^2\tau+\nu\cos\tau,(\varsigma,\nu)\in(-\infty,1]\times(-\infty,1]\\
&\backslash(0,1]\times(0,1],\tau\in\chi,\qquad(6)\\
&x(\varsigma,0,\tau)=\varsigma^2\sin^2\tau,\varsigma\in[0,1],x(0,\nu,\tau)=\nu\cos\tau,\\
&\nu\in[0,1],\tau\in\chi.\qquad(7)
\end{aligned}
$$

Let $\mathfrak{Y}:=[0,1]\times\{0\}\cup\{0\}\times[0,1]$, and $x:(-\infty,1]\times(-\infty,1]\to\mathbb{R}$ where $x_{(\varsigma,\nu)}\in\mathfrak{K}$ for $(\varsigma,\nu)\in\mathfrak{Y}$.
Thus, we demonstrate that, there is a two functions $\gamma,\sigma:\mathbb{R}_+\to\mathbb{R}_+$ such that $\|x_{(\varsigma,\nu)}\|_{\mathfrak{K}}\leq\gamma(\varsigma,\nu)\sup\{|x(\rho,k)|:$

$(\rho,k)\in[0,\varsigma]\times[0,\nu]\}+\sigma(\varsigma,\nu)\|\varpi\|_{\mathfrak{K}}$.
If $\varsigma+\varepsilon\leq 0,\nu+\lambda\leq 0$, we get, $\|x_{(\varsigma,\nu)}\|_{\mathfrak{K}}=\sup\{|x(\rho,k)|:(\rho,k)\in(-\infty,0]\times(-\infty,0]\}$,
and if $\varsigma+\varepsilon\geq 0,\nu+\lambda\geq 0$, we get, $\|x_{(\varsigma,\nu)}\|_{\mathfrak{K}}=\sup\{|x(\rho,k)|:(\rho,k)\in[0,\varsigma]\times[0,\nu]\}$.
Then, $\forall(\varsigma+\varepsilon,\varsigma+\lambda)\in[0,1]\times[0,1]$, one can get,
$\|x_{(\varsigma,\nu)}\|_{\mathfrak{K}}=\sup\{|x(\rho,k)|:(\rho,k)\in(-\infty,0]\times(-\infty,0]\}$
$+\sup\{|x(\rho,k)|:(\rho,k)\in[0,\varsigma]\times[0,\nu]\}$.
Then, $\|x_{(\varsigma,\nu)}\|_{\mathfrak{K}}=\sup\{\|x(\rho,k)\|_{\mathfrak{K}}:(\rho,k)\in\mathfrak{Y}\}$
$+\sup\{|x(\rho,k)|:(\rho,k)\in[0,\varsigma]\times[0,\nu]\}$.
$(\mathfrak{K},\|\cdot\|_{\mathfrak{K}})$ be seminormed linear space. We conclude that $\mathfrak{K}$ is a phase space. Set $|\psi(\varsigma,\nu,x_{(\varsigma,\nu)})|\leq\frac{1}{5}+\frac{\|x_{(\varsigma,\nu)}\|}{3+\|x_{(\varsigma,\nu)}\|},(\varsigma,\nu)\in[0,1]\times[0,1]$.
The functions $\tau\mapsto\varpi(\varsigma,0,\tau)=\varsigma^2\sin^2\tau,\tau\mapsto\varpi(0,\nu,\tau)=\nu\cos\tau$ and $\tau\mapsto\varpi(\varsigma,\nu,\tau)=\varsigma^2\sin^2\tau+\nu\cos\tau$ are measurable and bounded with $|\varpi_1(\varsigma,0,\tau)|\leq 1,\varpi_2(0,\nu,\tau)\leq 1$, Thus, $H1$ is verified. As expected, $(\varsigma,\nu,\tau)\mapsto\psi(\varsigma,\nu,x_{(\varsigma,\nu)},\tau)$ is jointly continuous for all $x\in\mathfrak{K}$, thus, jointly measurable for all $x\in\mathfrak{K}$. $x\mapsto\psi(\varsigma,\nu,x_{(\varsigma,\nu)},\tau)$ is continuous for all $(\varsigma,\nu)\in\Theta$ and $\tau\in\chi$. so the function ψ is carath*é*odory on $[0,1]\times[0,1]\times\mathfrak{K}\times\chi$.
For each $x_{(\varsigma,\nu)}\in\mathfrak{K}$. $(\varsigma,\nu)\in[0,1]\times[0,1]$ and $\tau\in\chi$, we have, $|\psi(\varsigma,\nu,x_{(\varsigma,\nu)})|\leq\frac{1}{5}+\frac{1}{3}\|x\|_{\mathfrak{K}}$. Thus, $H5$ is fulfilled with $\sigma_1(\varsigma,\nu,\tau)=\sigma_1^*=\frac{1}{5},\sigma_2(\varsigma,\nu,\tau)=\sigma_2^*=\frac{1}{3}$.
Also, $H6$ is verified. We have to verify that $\aleph<1$ with $\varsigma=\nu=1$. For each $(\xi_1,\xi_2)\in(0,1]\times(0,1],\eta\approx 0.3,\ell_0\approx 0.2$, we get, $\aleph\approx 0.42<1$.
Consequently, Theorem 1 is satisfied. Hence, the system $(5)-(7)$ has a random solution and it is controllable on $(-\infty,1]\times(-\infty,1]$.

V. CONCLUSION

This work investigated the controllability properties of fractional partial random delay differential equations in Banach space. The study drew major findings using Monch's fixed point theorem, fractional calculus principles, noncompactness metrics. Further, some new works can be done by modifying the functional spaces and derivatives or by expanding the study to trajectory controllability by including concepts such as deviated arguments, external noises, etc. This framework not only enhances theoretical understanding of PRDE but also provides practical approaches.

REFERENCES

[1] Miller, K. S., Ross, B. An Introduction to the Fractional Calculus and Differential Equations, *John Wiley*, New York,1993.

[2] Hilfer,R. Applications of Fractional Calculus in Physics, *World Scientific*, Singapore, 985-999, 2000.

[3] Kilbas, A.A., Srivastava, H.M., -Trujillo, J.J. Theory and Applications of Fractional Differential Equations, *North-Holland mathematics studies*, Amsterdam, 198, 2006.

[4] Podlubny I. Fractional Differential Equations, *mathematics in science and engineering.* Academic Press, San Diego, 198,1998.

[5] Lv ,J. and Yang, X. Approximate controllability of Hilfer fractional differential equations. *Mathematical Methods in the Applied Sciences*, 43(1), 242-254, 2020.

[6] Abbas, S., Benchohra, M., Cabada, A. Partial neutral functional integro-differential equations of fractional order with delay. *Boundary Value Problem*, 2012, 15, 2012.

[7] H. M. Ahmed, M. M. El-Borai, A. O. El Bab, and M. E. Ramadan. Approximate controllability of non instantaneous impulsive Hilfer fractional integro-differential equations with fractional brownian motion. *Boundary Value Problems*,2020(1), 1-25, 2020.

[8] Hussain, S., Sarwar, M., Abodayeh, K., Promsakon, C., & Sitthiwirattham, T. (2024). Controllability of Hilfer fractional neutral impulsive stochastic delayed differential equations with nonlocal conditions. *Chaos, Solitons and Fractals*, 183, 114876.

[9] Engl HW. A general stochastic fixed-point theorem for continuous random operators on stochastic domains. *Journal of Mathematical Analysis and Applications*, 66,31-220, 1978.

[10] Bharucha-Reid, AT. Random integral equations. *New York: Academic Press*; 1972.

[11] Benchohra, M., Bouazzaoui, F., Karapinar, E., Salim A. Controllability of Second Order Functional Random Differential Equations with Delay. *Mathematics*, 10(7),1120, 2022.

[12] Heris, A., Salim, A., Benchohra, M., & Karapinar, E. Fractional Partial Random Differential Equation with infinite delay. *Results in Physics*, 37, 105557, 2022.

[13] Hale J, Kato J. Phase space for retarded equations with infinite delay. *Funkcial Ekvac* 1978;21:11-41.

[14] Hino, Y., Murakami,S.,Naito T. Functional differential equations with infinite delay. *Lecture notes in mathematics,* vol. 1473, Berlin: Springer-Verlag; 1991.

[15] Abbas. S., Benchohra. M.,*N'Guérékata* GM. Topics in fractional differential equations. New York: Springer; 2012.

[16] Abbas S, Benchohra M. Darboux problem for perturbed partial differential equations of fractional order with finite delay. *Nonlinear Anal Hybrid Syst,* 3, 597-604, 2009.

[17] Banas, J.,Goebel, K. Measures of Noncompactness in Banach space; *Marcel Dekker: New York,* NY, USA, 1980.

[18] Guo,D.J., Lakshmikantham, V., Liu, X. Nonlinear Integral Equations in Abstract Spaces, *Kluwer Academic Publishers: Dordrecht*, The Netherlands, 1996.

[19] Itoh S. Random fixed point theorems with applications to random differential equations in Banach spaces. *Journal of Mathematical Analysis and Applications*, 67, 73-261, 1979.