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Design of Feedback Control for Three-Time-Scale System using Multirate Sampling of States

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Abstract— **This paper presents the design of control for a linear time-invariant system with three different time scales using multirate state feedback (MRSF). The term "multirate sampling" in this context refers to the act of sampling the slow, fast, and fastest states at varying time intervals. Initially, the original continuous three-time-scale system is transformed into a block triangular form. Subsequently, it is discretized in a step-by-step manner, and feedback controls are simultaneously devised at different sampling rates to create a unified MRSF control. It is established that this derived MRSF control stabilizes the entire system. Additionally, a sequential three-stage observer is recommended for estimating the transformed slow, fast, and fastest states, which is crucial for implementing MRSF. Lastly, the design is illustrated through a numerical example.**

I. INTRODUCTION

The use of feedback control is known to enhance system stability and performance. With advancements in computing technology, the study of discrete-time system stabilization using state feedback has gained momentum. However, designing feedback control for multi-time-scale systems poses challenges due to eigenvalue clustering [1]. Feedback control is often designed in multiple stages to address this issue, for example, two-stage design [2], [3] and three-stage design [4]. In a three-time-scale system, fast and fastest states play a vital role during a transient period, and thereafter slow states come into the picture. Sampling all states at a single rate can lead to performance degradation. To overcome this problem, a multirate sampling of states is proposed [5].

Previous works [6]–[8] have established the conceptual background and recommended the optimal regulators designs using multirate sampling. However, no work is reported on a multirate sampling of states for the three-time-scale system using the block-diagonalization approach. Recently, in [9], [10], MRSF control for a two-time-scale system is presented. In this, a sequential two-stage observer is formulated for state estimation.

This paper not only provides the extension to [9], [10] for a three-time-scale system but also details the selection of sampling periods for three different subsystems. This approach reduces computations and signal processing time while improving the performance of the system. The effectiveness of the proposed design is demonstrated through a numerical example. The paper is arranged into sections. Section II covers the proposed MRSF control design methodology. Section III presents state estimation using the sequential three-stage observer. Section IV demonstrates the applicability of the controllers to the example, and concluding remarks are given in Section V.

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II. PROPOSED MRSF CONTROL DESIGN

A. System Explanation

Let the continuous-time system be represented as

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}, \tag{1}$$

$$\mathbf{y} = \mathbf{C}\mathbf{z}, \tag{2}$$

where $\mathbf{z} \in \Re^n$ is state, $\mathbf{u} \in \Re^m$ is input and $\mathbf{y} \in \Re^p$ is output vectors. If the system (1)-(2) exhibits three-time-scale behaviour, i.e., n_1 slow, n_2 fast, and n_3 fastest eigenvalues, where $n_1 + n_2 + n_3 = n$, then one can write

$$\begin{bmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \\ \dot{\mathbf{z}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \frac{\mathbf{A}_{21}}{\varepsilon} & \frac{\mathbf{A}_{22}}{\varepsilon} & \frac{\mathbf{A}_{23}}{\varepsilon} \\ \frac{\mathbf{A}_{31}}{\mu} & \frac{\mathbf{A}_{32}}{\mu} & \frac{\mathbf{A}_{33}}{\mu} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \frac{\mathbf{B}_2}{\varepsilon} \\ \frac{\mathbf{B}_3}{\mu} \end{bmatrix} \mathbf{u}, \tag{3}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 & \mathbf{C}_3 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1^T & \mathbf{z}_2^T & \mathbf{z}_3^T \end{bmatrix}^T, \tag{4}$$

where $\mathbf{z}_1 \in \Re^{n_1}$, $\mathbf{z}_2 \in \Re^{n_2}$, $\mathbf{z}_3 \in \Re^{n_3}$ and $\mathbf{A}_{ij}$, $\mathbf{B}_i$ and $\mathbf{C}_i$ are of appropriate sizes. Parameters ε and μ are speed ratios of the slow-to-fast-to-fastest states ($\mu \ll \varepsilon \ll 1$).

Assumption 1: The system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ satisfies controllability and observability properties [11].

Assumption 2: $\mathbf{A}_{22}$ and $\mathbf{A}_{33}$ are invertible [12].

Let N_1 and N_2 be integers, such that $N_1 \leq \frac{1}{\varepsilon}$ and $N_2 \leq \frac{\varepsilon}{\mu}$ and τ, Δ and δ are sampling periods of slow, fast and fastest states respectively, with $\tau > \Delta > \delta$, then

$$\tau = N_1\Delta = N_1N_2\delta. \tag{5}$$

In general, parameters ε and μ may not be present in the system all the time, i.e. singularly perturbed nature may not be visible directly. Still, the system can have multiple eigenvalue groups. In fact, singularly perturbed systems are a class of multi-time-scale systems and the former can be expressed in the latter form. Interestingly, for an appropriately selected range of sampling periods,

the discrete-time system matching to the continuous-time three-time-scale system (3), retains the three-time-scale property [4]. That's why the sampling periods δ, Δ, and τ should be selected properly. Let the sampling starts at a $k\tau$. Here, slow states are recorded at each $k\tau$, where $k = 0, 1, 2, ...$, whereas the fast states are recorded at each $k\Delta = k\left(\frac{\tau}{N_1}\right)$ instant and the fastest states are recorded at each $k\delta = k\left(\frac{\tau}{N_1 N_2}\right)$ instant. Other way round, in τ sampling period of slow states, fast states are sampled N_1 times at every $h\Delta$, where $h = 0, 1, 2, ..., (N_1 - 1)$, and in Δ sampling interval of fast states, fastest states are sampled N_2 times at every $l\delta$, where $l = 0, 1, 2, ..., (N_2 - 1)$. The overall control is

$$\mathbf{u}(t) = \mathbf{u}_s(k\tau) + \mathbf{u}_{f1}(k\tau + h\Delta) + \mathbf{u}_{f2}(k\tau + h\Delta + l\delta), \quad (6)$$

where $\mathbf{u}_s(k\tau)$, $\mathbf{u}_{f1}(k\tau + h\Delta)$ and $\mathbf{u}_{f2}(k\tau + h\Delta + l\delta)$ are piece-wise constant during

$$\begin{cases} \mathbf{u}_s(k\tau) & k\tau \le t < (k+1)\tau, \\ \mathbf{u}_{f1}(k\tau + h\Delta) & k\tau + h\Delta \le t < k\tau + (h+1)\Delta, \\ \mathbf{u}_{f2}(k\tau + h\Delta + l\delta) & k\tau + h\Delta + l\delta \le t < k\tau + h\Delta + (l+1)\delta, \end{cases}$$

respectively.

B. Design of MRSF for Three-Time-Scale System

To start with, the block triangular form of (3) is obtained as

$$\begin{bmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_{fs} \\ \dot{\mathbf{z}}_{f2} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_s & \bar{\mathbf{A}}_{12} & \mathbf{A}_{13} \\ \mathbf{0} & \frac{\mathbf{A}_{f1}}{\varepsilon} & \frac{\bar{\mathbf{A}}_{23}}{\varepsilon} \\ \mathbf{0} & \mathbf{0} & \frac{\mathbf{A}_{f2}}{\mu} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_{fs} \\ \mathbf{z}_{f2} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \frac{\bar{\mathbf{B}}_{f1}}{\varepsilon} \\ \frac{\mathbf{B}_{f2}}{\mu} \end{bmatrix} \mathbf{u}. \quad (7)$$

In this, $max|\text{Re}\{\lambda(\mathbf{A}_s)\}| \ll min\left|\text{Re}\left\{\lambda\left(\frac{\mathbf{A}_{f1}}{\varepsilon}\right)\right\}\right|$ and $max\left|\text{Re}\left\{\lambda\left(\frac{\mathbf{A}_{f1}}{\varepsilon}\right)\right\}\right| \ll min\left|\text{Re}\left\{\lambda\left(\frac{\mathbf{A}_{f2}}{\mu}\right)\right\}\right|$, where $\lambda(.)$ is eigenvalue. The system (7) is gained by applying

$$\begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_{fs} \\ \mathbf{z}_{f2} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{n1} & \mathbf{0} & \mathbf{0} \\ \mathbf{P}_{21} & \mathbf{I}_{n2} & \mathbf{0} \\ \mathbf{P}_{32}\mathbf{P}_{21} + \mathbf{P}_{31} & \mathbf{P}_{32} & \mathbf{I}_{n3} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \end{bmatrix}, \quad (8)$$

to (3), in which $\mathbf{I}_n$ is an identity matrix of order n and matrices $\mathbf{P}_{21}$ and $\mathbf{P}_{31}$ satisfy following algebraic equations

$$\left.\begin{array}{rl} \varepsilon\mathbf{P}_{21}\mathbf{A}_{11} + \mathbf{A}_{21} - \mathbf{A}_{f22}\mathbf{P}_{21} - \bar{\mathbf{A}}_{23}\mathbf{P}_{31} & = 0, \\ \mu\mathbf{P}_{31}\mathbf{A}_{11} + \mathbf{A}_{31} - \mathbf{A}_{f32}\mathbf{P}_{21} - \mathbf{A}_{f33}\mathbf{P}_{31} & = 0, \end{array}\right\} \quad (9)$$

where $\mathbf{A}_s = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{P}_{21} - \mathbf{A}_{13}\mathbf{P}_{31}$, $\mathbf{A}_{f22} = \varepsilon\mathbf{P}_{21}\mathbf{A}_{12} + \mathbf{A}_{22}$, $\bar{\mathbf{A}}_{23} = \varepsilon\mathbf{P}_{21}\mathbf{A}_{13} + \mathbf{A}_{23}$, $\mathbf{A}_{f32} = \mu\mathbf{P}_{31}\mathbf{A}_{12} + \mathbf{A}_{32}$, $\mathbf{A}_{f33} = \mu\mathbf{P}_{31}\mathbf{A}_{13} + \mathbf{A}_{33}$, $\bar{\mathbf{B}}_{f1} = \varepsilon\mathbf{P}_{21}\mathbf{B}_1 + \mathbf{B}_2$ and $\bar{\mathbf{B}}_{f2} = \mu\mathbf{P}_{31}\mathbf{B}_1 + \mathbf{B}_3$. Matrix $\mathbf{P}_{32}$ satisfies

$$\frac{\mu}{\varepsilon}\mathbf{P}_{32}\mathbf{A}_{f22} + \mathbf{A}_{f32} - \left(\frac{\mu}{\varepsilon}\mathbf{P}_{32}\bar{\mathbf{A}}_{23} + \mathbf{A}_{f33}\right)\mathbf{P}_{32} = 0. \quad (10)$$

with $\bar{\mathbf{A}}_{12} = \mathbf{A}_{12} - \mathbf{A}_{13}\mathbf{P}_{32}$, $\mathbf{A}_{f1} = \mathbf{A}_{f22} - \bar{\mathbf{A}}_{23}\mathbf{P}_{32}$, $\mathbf{A}_{f2} = \frac{\mu}{\varepsilon}\mathbf{P}_{32}\bar{\mathbf{A}}_{23} + \mathbf{A}_{f33}$ and $\mathbf{B}_{f2} = \frac{\mu}{\varepsilon}\mathbf{P}_{32}\bar{\mathbf{B}}_{f1} + \bar{\mathbf{B}}_{f2}$. As per Assumption 2, the equations in (9) can be solved for a very small value of ε. For that putting $\varepsilon = 0$ in (9) provides

$$\mathbf{A}_{21} - \mathbf{A}_{f22}\mathbf{P}_{21}^{(0)} - \bar{\mathbf{A}}_{23}\mathbf{P}_{31}^{(0)} = 0, \quad (11)$$

$$\mathbf{A}_{31} - \mathbf{A}_{f32}\mathbf{P}_{21}^{(0)} - \mathbf{A}_{f33}\mathbf{P}_{31}^{(0)} = 0. \quad (12)$$

From (11), $\mathbf{P}_{21}^{(0)} = \mathbf{A}_{f22}^{-1}(\mathbf{A}_{21} - \bar{\mathbf{A}}_{23}\mathbf{P}_{31}^{(0)}) \Longrightarrow \mathbf{P}_{21}^{(0)} = \mathbf{A}_{22}^{-1}(\mathbf{A}_{21} - \mathbf{A}_{23}\mathbf{P}_{31}^{(0)})$ and $\mathbf{P}_{21}^{(0)} = O(1)$ and $\mathbf{P}_{21} = \mathbf{P}_{21}^{(0)} + O(\varepsilon)$. Substituting $\mathbf{P}_{21}^{(0)}$ in (12) provides $\mathbf{P}_{31}^{(0)} = (\mathbf{A}_{33} - \mathbf{A}_{32}\mathbf{A}_{22}^{-1}\mathbf{A}_{23})^{-1}(\mathbf{A}_{31} - \mathbf{A}_{32}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})$ and $\mathbf{P}_{31}^{(0)} = O(1)$ and $\mathbf{P}_{31} = \mathbf{P}_{31}^{(0)} + O(\varepsilon)$. Since $\mathbf{A}_{f22} = \mathbf{A}_{22} + O(\varepsilon)$ and assuming Assumption 2 is satisfied, existence of $\mathbf{A}_{f22}^{-1}$ is ensured. Next, to get $\mathbf{P}_{31}$, Assumption 3 is given.

Assumption 3: $(\mathbf{A}_{33} - \mathbf{A}_{32}\mathbf{A}_{22}^{-1}\mathbf{A}_{23})$ is invertible.

Similarly, unique solution of (10) for $\mathbf{P}_{32}$ exists for small value of μ with Assumption 2, i.e. for $\mu = 0$, $\mathbf{A}_{f33}\mathbf{P}_{32}^{(0)} = \mathbf{A}_{f32} \Longrightarrow \mathbf{P}_{32}^{(0)} = \mathbf{A}_{33}^{-1}\mathbf{A}_{32} + O(\mu)$. Having found $\mathbf{P}_{21}^{(0)}$, $\mathbf{P}_{31}^{(0)}$ and $\mathbf{P}_{32}^{(0)}$ algebraic equations (9) and (10) can be operated as given in [13]. As per Assumption 1, the system (7) is controllable where the fastest subsystem is completely decoupled from the slow and fast subsystems. Here, as per Assumption 1, controllability property of $\left(\frac{\mathbf{A}_{f2}}{\mu}, \frac{\mathbf{B}_{f2}}{\mu}\right)$ is conserved. Now, discretizing (7) with δ

$$\begin{bmatrix} \mathbf{z}_{1,l+1} \\ \mathbf{z}_{fs,l+1} \\ \mathbf{z}_{f2,l+1} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_s & \mathbf{\Phi}_{12} & \mathbf{\Phi}_{13} \\ \mathbf{0} & \mathbf{\Phi}_{f1} & \mathbf{\Phi}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{f2} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1,l} \\ \mathbf{z}_{fs,l} \\ \mathbf{z}_{f2,l} \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma}_1 \\ \bar{\mathbf{\Gamma}}_{f1} \\ \mathbf{\Gamma}_{f2} \end{bmatrix} \mathbf{u}_l. \quad (13)$$

In this, let

$$\bar{\mathbf{\Phi}} = \begin{bmatrix} \mathbf{\Phi}_s & \mathbf{\Phi}_{12} & \mathbf{\Phi}_{13} \\ \mathbf{0} & \mathbf{\Phi}_{f1} & \mathbf{\Phi}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{f2} \end{bmatrix} = e^{\bar{\mathbf{A}}\delta}, \bar{\mathbf{\Gamma}} = \begin{bmatrix} \mathbf{\Gamma}_1 \\ \bar{\mathbf{\Gamma}}_{f1} \\ \mathbf{\Gamma}_{f2} \end{bmatrix} = \int_0^\delta e^{\bar{\mathbf{A}}\eta}\bar{\mathbf{B}}d\eta,$$

with

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_s & \bar{\mathbf{A}}_{12} & \mathbf{A}_{13} \\ \mathbf{0} & \frac{\mathbf{A}_{f1}}{\varepsilon} & \frac{\bar{\mathbf{A}}_{23}}{\varepsilon} \\ \mathbf{0} & \mathbf{0} & \frac{\mathbf{A}_{f2}}{\mu} \end{bmatrix} \text{ and } \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ \frac{\bar{\mathbf{B}}_{f1}}{\varepsilon} \\ \frac{\mathbf{B}_{f2}}{\mu} \end{bmatrix}.$$

If the original continuous-time system is controllable and has only real eigenvalues, then its discretized version, for $\delta > 0$, is also controllable. However, if the system has complex conjugate eigenvalues then δ should be appropriately selected [11]. Hence, Assumption 4 is needed.

Assumption 4: The system $(\bar{\mathbf{\Phi}}, \bar{\mathbf{\Gamma}})$ is controllable.

Therefore, the subsystem $(\mathbf{\Phi}_{f2}, \mathbf{\Gamma}_{f2})$ is also controllable. And so, in the first stage, feedback control

$$\mathbf{u}_l = -\mathbf{F}_{f2}\mathbf{z}_{f2,l} + \mathbf{u}_{f1,l}, \quad (14)$$

is designed and applied to (13) as

$$\begin{bmatrix} \mathbf{z}_{1,l+1} \\ \mathbf{z}_{fs,l+1} \\ \mathbf{z}_{f2,l+1} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_s & \mathbf{\Phi}_{12} & \mathbf{\Phi}_{13} - \mathbf{\Gamma}_1\mathbf{F}_{f2} \\ \mathbf{0} & \mathbf{\Phi}_{f1} & \mathbf{\Phi}_{23} - \bar{\mathbf{\Gamma}}_{f1}\mathbf{F}_{f2} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{f2} - \mathbf{\Gamma}_{f2}\mathbf{F}_{f2} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1,l} \\ \mathbf{z}_{fs,l} \\ \mathbf{z}_{f2,l} \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma}_1 \\ \bar{\mathbf{\Gamma}}_{f1} \\ \mathbf{\Gamma}_{f2} \end{bmatrix} \mathbf{u}_{f1,l}, \quad (15)$$

for getting $\lambda(\mathbf{\Phi}_{f2} - \mathbf{\Gamma}_{f2}\mathbf{F}_{f2}) = \lambda_{\delta f2}^{desired}$, system (15) is

$$\begin{bmatrix} \mathbf{z}_{1,l+1} \\ \mathbf{z}_{f1,l+1} \\ \mathbf{z}_{f2,l+1} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_s & \mathbf{\Phi}_{12} & \bar{\mathbf{\Phi}}_{13} \\ \mathbf{0} & \mathbf{\Phi}_{f1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{\Phi}}_{f2} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1,l} \\ \mathbf{z}_{f1,l} \\ \mathbf{z}_{f2,l} \end{bmatrix} + \begin{bmatrix} \mathbf{\Gamma}_1 \\ \mathbf{\Gamma}_{f1} \\ \mathbf{\Gamma}_{f2} \end{bmatrix} \mathbf{u}_{f1,l}, \quad (16)$$

to separated fast subsystems, where $\bar{\mathbf{\Phi}}_{13} = \mathbf{\Phi}_{13} - \mathbf{\Gamma}_1\mathbf{F}_{f2}$, $\bar{\mathbf{\Phi}}_{23} = \mathbf{\Phi}_{23} - \bar{\mathbf{\Gamma}}_{f1}\mathbf{F}_{f2}$ and $\bar{\mathbf{\Phi}}_{f2} = \mathbf{\Phi}_{f2} - \mathbf{\Gamma}_{f2}\mathbf{F}_{f2}$. This is done by applying

$$\mathbf{z}_{f1,l} = \mathbf{z}_{fs,l} + \mathbf{M}_{23}\mathbf{z}_{f2,l}, \quad (17)$$

to (15). In this, the $\mathbf{M}_{23}$ is obtained using Assumption 5

$$-\mathbf{\Phi}_{f1}\mathbf{M}_{23}+\bar{\mathbf{\Phi}}_{23}+\mathbf{M}_{23}\bar{\mathbf{\Phi}}_{f2}=0. \tag{18}$$

Assumption 5: $\lambda(\mathbf{\Phi}_{f1})\neq\lambda(\bar{\mathbf{\Phi}}_{f2})$.

Manipulating (15) and (17) gives $\bar{\mathbf{\Phi}}_{13}=-\mathbf{\Phi}_{12}\mathbf{M}_{23}+\bar{\mathbf{\Phi}}_{13}$ and $\bar{\mathbf{\Gamma}}_{f1}=\bar{\mathbf{\Gamma}}_{f1}+\mathbf{M}_{23}\mathbf{\Gamma}_{f2}$. The system (16) has a time period of δ seconds. If $\Delta=N_2\delta$, system consistent with Δ time period can be constructed using (16) as

$$\begin{bmatrix}\mathbf{z}_{1,h+1}\\ \mathbf{z}_{f1,h+1}\\ \mathbf{z}_{f2,h+1}\end{bmatrix}=\begin{bmatrix}\mathbf{\Phi}_{\Delta s} & \mathbf{\Phi}_{\Delta 12} & \mathbf{\Phi}_{\Delta 13}\\ \mathbf{0} & \mathbf{\Phi}_{\Delta f1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{\Phi}_{\Delta f2}\end{bmatrix}\begin{bmatrix}\mathbf{z}_{1,h}\\ \mathbf{z}_{f1,h}\\ \mathbf{z}_{f2,h}\end{bmatrix}+\begin{bmatrix}\mathbf{\Gamma}_{\Delta 1}\\ \mathbf{\Gamma}_{\Delta f1}\\ \mathbf{\Gamma}_{\Delta f2}\end{bmatrix}\mathbf{u}_{f1,h}. \tag{19}$$

The system and input matrices of (16) and (19) are related as $\mathbf{\Phi}_{\Delta}=\mathbf{\Phi}_{\delta}^{N_2}$ and $\mathbf{\Gamma}_{\Delta}=\sum_{i=0}^{N_2-1}\mathbf{\Phi}_{\delta}^{i}\mathbf{\Gamma}_{\delta}$. Using this relation, submatrices in (19) are calculated as

$$\left.\begin{array}{rl}\mathbf{\Phi}_{\Delta s}= & \mathbf{\Phi}_{s}^{N_2},\mathbf{\Phi}_{\Delta f1}=\mathbf{\Phi}_{f1}^{N_2},\mathbf{\Phi}_{\Delta f2}=\bar{\mathbf{\Phi}}_{f2}^{N_2},\\ \mathbf{\Phi}_{\Delta 12}= & \sum_{i=0}^{N_2-1}\left(\mathbf{\Phi}_{s}^{N_2-1-i}\mathbf{\Phi}_{12}\mathbf{\Phi}_{f1}^{i}\right),\\ \mathbf{\Phi}_{\Delta 13}= & \sum_{i=0}^{N_2-1}\left(\mathbf{\Phi}_{s}^{N_2-1-i}\tilde{\mathbf{\Phi}}_{13}\bar{\mathbf{\Phi}}_{f2}^{i}\right),\\ \mathbf{\Gamma}_{\Delta 1}= & \sum_{i=0}^{N_2-1}\mathbf{\Phi}_{s}^{i}\mathbf{\Gamma}_{1}+\sum_{k=1}^{N_2-1}\left\{\sum_{i=0}^{k-1}\mathbf{\Phi}_{s}^{k-1-i}\right.\\ & \left.\left(\mathbf{\Phi}_{12}\mathbf{\Phi}_{f1}^{i}\mathbf{\Gamma}_{f1}+\tilde{\mathbf{\Phi}}_{13}\bar{\mathbf{\Phi}}_{f2}^{i}\mathbf{\Gamma}_{f2}\right)\right\},\\ \mathbf{\Gamma}_{\Delta f1}= & \sum_{i=0}^{N_2-1}\mathbf{\Phi}_{f1}^{i}\mathbf{\Gamma}_{f1},\mathbf{\Gamma}_{\Delta f2}=\sum_{i=0}^{N_2-1}\bar{\mathbf{\Phi}}_{f2}^{i}\mathbf{\Gamma}_{f2}.\end{array}\right\} \tag{20}$$

An input $\mathbf{u}_{f1,l}$ in (14), at δ sampling interval, is now changed to the $\mathbf{u}_{f1,h}$ at Δ sampling interval in (19). In the second stage, the input is

$$\mathbf{u}_{f1,h}=-\mathbf{F}_{f1}\mathbf{z}_{f1,h}+\mathbf{u}_{s,h}, \tag{21}$$

where $\mathbf{u}_{s,h}$ is a slow subsystem input at Δ sampling period, which later will be represented at τ sampling interval. The input (21) is passed to (19) to have

$$\begin{bmatrix}\mathbf{z}_{1,h+1}\\ \mathbf{z}_{f1,h+1}\\ \mathbf{z}_{f2,h+1}\end{bmatrix}=\begin{bmatrix}\mathbf{\Phi}_{\Delta s} & \mathbf{\Phi}_{\Delta 12}-\mathbf{\Gamma}_{\Delta 1}\mathbf{F}_{f1} & \mathbf{\Phi}_{\Delta 13}\\ \mathbf{0} & \mathbf{\Phi}_{\Delta f1}-\mathbf{\Gamma}_{\Delta f1}\mathbf{F}_{f1} & \mathbf{0}\\ \mathbf{0} & -\mathbf{\Gamma}_{\Delta f2}\mathbf{F}_{f1} & \mathbf{\Phi}_{\Delta f2}\end{bmatrix}\begin{bmatrix}\mathbf{z}_{1,h}\\ \mathbf{z}_{f1,h}\\ \mathbf{z}_{f2,h}\end{bmatrix}+\begin{bmatrix}\mathbf{\Gamma}_{\Delta 1}\\ \mathbf{\Gamma}_{\Delta f1}\\ \mathbf{\Gamma}_{\Delta f2}\end{bmatrix}\mathbf{u}_{s,h}, \tag{22}$$

so that $\lambda(\mathbf{\Phi}_{\Delta f1}-\mathbf{\Gamma}_{\Delta f1}\mathbf{F}_{f1})=\lambda_{\Delta f1}^{desired}$. Defining $\bar{\mathbf{\Phi}}_{\Delta 12}=\mathbf{\Phi}_{\Delta 12}-\mathbf{\Gamma}_{\Delta 1}\mathbf{F}_{f1}$, $\bar{\mathbf{\Phi}}_{\Delta f1}=\mathbf{\Phi}_{\Delta f1}-\mathbf{\Gamma}_{\Delta f1}\mathbf{F}_{f1}$ and $\bar{\mathbf{\Phi}}_{\Delta 32}=-\mathbf{\Gamma}_{\Delta f2}\mathbf{F}_{f1}$, system (22) is changed to

$$\begin{bmatrix}\mathbf{z}_{s,h+1}\\ \mathbf{z}_{f1,h+1}\\ \mathbf{z}_{f2,h+1}\end{bmatrix}=\begin{bmatrix}\mathbf{\Phi}_{\Delta s} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \bar{\mathbf{\Phi}}_{\Delta f1} & \mathbf{0}\\ \mathbf{0} & \bar{\mathbf{\Phi}}_{\Delta 32} & \mathbf{\Phi}_{\Delta f2}\end{bmatrix}\begin{bmatrix}\mathbf{z}_{s,h}\\ \mathbf{z}_{f1,h}\\ \mathbf{z}_{f2,h}\end{bmatrix}+\begin{bmatrix}\mathbf{\Gamma}_{\Delta s}\\ \mathbf{\Gamma}_{\Delta f1}\\ \mathbf{\Gamma}_{\Delta f2}\end{bmatrix}\mathbf{u}_{s,h}. \tag{23}$$

for decoupling slow subsystem by applying

$$\mathbf{z}_{s,h}=\mathbf{z}_{1,h}+\mathbf{M}_{12}\mathbf{z}_{f1,h}+\mathbf{M}_{13}\mathbf{z}_{f2,h}, \tag{24}$$

to (22). Matrices $\mathbf{M}_{12}$ and $\mathbf{M}_{13}$ are found from

$$-\mathbf{\Phi}_{\Delta s}\mathbf{M}_{13}+\mathbf{\Phi}_{\Delta 13}+\mathbf{M}_{13}\mathbf{\Phi}_{\Delta f2}=0, \tag{25}$$

$$-\mathbf{\Phi}_{\Delta s}\mathbf{M}_{12}+\bar{\mathbf{\Phi}}_{\Delta 12}+\mathbf{M}_{12}\bar{\mathbf{\Phi}}_{\Delta f1}+\mathbf{M}_{13}\bar{\mathbf{\Phi}}_{\Delta 32}=0. \tag{26}$$

Equations (25) and (26) can be solved independently, similar to (18). First, (25) is solved for $\mathbf{M}_{13}$, with

Assumption 6: $\lambda(\mathbf{\Phi}_{\Delta s})\neq\lambda(\mathbf{\Phi}_{\Delta f2})$.

Having obtained $\mathbf{M}_{13}$, (26) is solved to get $\mathbf{M}_{12}$, with

Assumption 7: $\lambda(\mathbf{\Phi}_{\Delta s})\neq\lambda(\bar{\mathbf{\Phi}}_{\Delta f1})$.

Application of (24) to (22) gives $\mathbf{\Gamma}_{\Delta s}=\mathbf{\Gamma}_{\Delta 1}+\mathbf{M}_{12}\mathbf{\Gamma}_{\Delta f1}+\mathbf{M}_{13}\mathbf{\Gamma}_{\Delta f2}$. Changing the states of (22) to (23) using (24) alters it from one form to another holding the controllability of the uncoupled slow subsystem $(\bar{\mathbf{\Phi}}_{\Delta s},\mathbf{\Gamma}_{\Delta s})$ as it is, by Assumption 4. The system (23) has a sampling period of Δ s. Using (5), an equivalent system at τ sampling interval can be developed from (23)

$$\begin{bmatrix}\mathbf{z}_{s,k+1}\\ \mathbf{z}_{f1,k+1}\\ \mathbf{z}_{f2,k+1}\end{bmatrix}=\begin{bmatrix}\mathbf{\Phi}_{\tau s} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Phi}_{\tau f1} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Phi}_{\tau 32} & \mathbf{\Phi}_{\tau f2}\end{bmatrix}\begin{bmatrix}\mathbf{z}_{s,k}\\ \mathbf{z}_{f1,k}\\ \mathbf{z}_{f2,k}\end{bmatrix}+\begin{bmatrix}\mathbf{\Gamma}_{\tau s}\\ \mathbf{\Gamma}_{\tau f1}\\ \mathbf{\Gamma}_{\tau f2}\end{bmatrix}\mathbf{u}_{s,k}. \tag{27}$$

Here, system and input matrices of systems (23) and (27) are related as $\mathbf{\Phi}_{\tau}=\mathbf{\Phi}_{\Delta_u}^{N_1}$ and $\mathbf{\Gamma}_{\tau}=\sum_{i=0}^{N_1-1}\mathbf{\Phi}_{\Delta_u}^{i}\mathbf{\Gamma}_{\Delta_u}$. As a result, submatrices in (27) are obtained as

$$\left.\begin{array}{rl}\mathbf{\Phi}_{\tau s}= & \mathbf{\Phi}_{\Delta s}^{N_1},\mathbf{\Phi}_{\tau f1}=\bar{\mathbf{\Phi}}_{\Delta f1}^{N_1},\mathbf{\Phi}_{\tau f2}=\mathbf{\Phi}_{\Delta f2}^{N_1},\\ \mathbf{\Phi}_{\tau 32}= & \sum_{i=0}^{N_1-1}(\mathbf{\Phi}_{\Delta f2}^{N_1-1-i}\bar{\mathbf{\Phi}}_{\Delta 32}\bar{\mathbf{\Phi}}_{\Delta f1}^{i}),\\ \mathbf{\Gamma}_{\tau s}= & \sum_{i=0}^{N_1-1}\mathbf{\Phi}_{\Delta s}^{i}\mathbf{\Gamma}_{\Delta s},\\ \mathbf{\Gamma}_{\tau f1}= & \sum_{i=0}^{N_1-1}\bar{\mathbf{\Phi}}_{\Delta f1}^{i}\mathbf{\Gamma}_{\Delta f1},\\ \mathbf{\Gamma}_{\tau f2}= & \sum_{i=0}^{N_1-1}\mathbf{\Phi}_{\Delta f2}^{i}\mathbf{\Gamma}_{\Delta f2}+\sum_{k=1}^{N_1-1}\\ & \left\{\sum_{i=0}^{k-1}\left(\mathbf{\Phi}_{\Delta f2}^{k-1-i}\bar{\mathbf{\Phi}}_{\Delta 32}\bar{\mathbf{\Phi}}_{\Delta f1}^{i}\right)\mathbf{\Gamma}_{\Delta f1}\right\}.\end{array}\right\} \tag{28}$$

Assuming that the change in sampling rate does not change controllability of $(\mathbf{\Phi}_{\tau s},\mathbf{\Gamma}_{\tau s})$, in third stage

$$\mathbf{u}_{s,k}=-\mathbf{F}_{s}\mathbf{z}_{s,k}, \tag{29}$$

is designed and applied to (27) to have

$$\begin{bmatrix}\mathbf{z}_{s,k+1}\\ \mathbf{z}_{f1,k+1}\\ \mathbf{z}_{f2,k+1}\end{bmatrix}=\begin{bmatrix}\mathbf{\Phi}_{\tau s}-\mathbf{\Gamma}_{\tau s}\mathbf{F}_{s} & \mathbf{0} & \mathbf{0}\\ -\mathbf{\Gamma}_{\tau f1}\mathbf{F}_{s} & \mathbf{\Phi}_{\tau f1} & \mathbf{0}\\ -\mathbf{\Gamma}_{\tau f2}\mathbf{F}_{s} & \mathbf{\Phi}_{\tau 32} & \mathbf{\Phi}_{\tau f2}\end{bmatrix}\begin{bmatrix}\mathbf{z}_{s,k}\\ \mathbf{z}_{f1,k}\\ \mathbf{z}_{f2,k}\end{bmatrix} \tag{30}$$

for $\lambda(\mathbf{\Phi}_{\tau s}-\mathbf{\Gamma}_{\tau s}\mathbf{F}_{s})=\lambda_{\tau s}^{desired}$. Thus, MRSF control is

$$\mathbf{u}(t)=-\mathbf{F}_{f2}\mathbf{z}_{f2,l}-\mathbf{F}_{f1}\mathbf{z}_{f1,h}-\mathbf{F}_{s}\mathbf{z}_{s,k}, \tag{31}$$

where

$$\begin{array}{ll}\mathbf{u}_{f2,l}=-\mathbf{F}_{f2}\mathbf{z}_{f2,l}, & k\tau+h\Delta+l\delta\leq t<k\tau+h\Delta+(l+1)\delta,\\ \mathbf{u}_{f1,h}=-\mathbf{F}_{f1}\mathbf{z}_{f1,h}, & k\tau+h\Delta\leq t<k\tau+(h+1)\Delta,\\ \mathbf{u}_{s,k}=-\mathbf{F}_{s}\mathbf{z}_{s,k}, & k\tau\leq t<(k+1)\tau.\end{array}$$

Lemma 1: Input (31), obtained by sampling fastest, fast, and slow states respectively at δ, Δ and τ periods, stabilizes (30).

Proof: In (30), eigenvalues are the union of eigenvalues of diagonal matrices $(\mathbf{\Phi}_{\tau s}-\mathbf{\Gamma}_{\tau s}\mathbf{F}_{s})$, $\mathbf{\Phi}_{\tau f1}$ and $\mathbf{\Phi}_{\tau f2}$. The subsystem $(\mathbf{\Phi}_{\tau s}-\mathbf{\Gamma}_{\tau s}\mathbf{F}_{s})$ is made stable with $\mathbf{F}_{s}$ so that the eigenvalues stay inside the unit circle near the circumference. Alternatively, subsystem $\mathbf{\Phi}_{\tau f1}$ is given by (28), in which the eigenvalues of $(\mathbf{\Phi}_{\Delta f1}-\mathbf{\Gamma}_{\Delta f1}\mathbf{F}_{f1})$ are made stable by the feedback gain $\mathbf{F}_{f1}$. Now, if $|\lambda(\mathbf{\Phi}_{\Delta f1}-\mathbf{\Gamma}_{\Delta f1}\mathbf{F}_{f1})|<1$, then

$$|\lambda((\mathbf{\Phi}_{\Delta f1}-\mathbf{\Gamma}_{\Delta f1}\mathbf{F}_{f1})^{N_1})|<1, \tag{32}$$

making $|\lambda(\mathbf{\Phi}_{\tau f1})|<1$. Similarly, the subsystem $\mathbf{\Phi}_{\tau f2}$ is given by (20) and (28), in which

$$|\lambda(\mathbf{\Phi}_{f2}-\mathbf{\Gamma}_{f2}\mathbf{F}_{f2})|<1. \tag{33}$$

If (33) is satisfied then $|\lambda((\mathbf{\Phi}_{f2}-\mathbf{\Gamma}_{f2}\mathbf{F}_{f2})^{N_2N_1})| < 1$, i.e. $|\lambda(\mathbf{\Phi}_{\tau f2})| < 1$. As a result, the control (31) achieves stabilization for the system (30). ■

The control (31) has slow, fast and fastest states, that cannot be observed and sampled directly for realizing (31). Hence, an observer is required to determine them individually, then they are sampled at different rates. Consequently, control (31) has a form

$$\mathbf{u}(t) = -\mathbf{F}_{f2}\hat{\mathbf{z}}_{f2,l} - \mathbf{F}_{f1}\hat{\mathbf{z}}_{f1,h} - \mathbf{F}_s\hat{\mathbf{z}}_{s,k}, \quad (34)$$

where $\hat{\mathbf{z}}_{f2}$, $\hat{\mathbf{z}}_{f1}$, and $\hat{\mathbf{z}}_s$ are the estimated states, which are sampled at δ, Δ and τ intervals respectively.

III. SEQUENTIAL OBSERVER DESIGN

Full-order observer for (3)-(4) can be written [11] as

$$\begin{bmatrix}\dot{\hat{\mathbf{z}}}_1\\ \dot{\hat{\mathbf{z}}}_2\\ \dot{\hat{\mathbf{z}}}_3\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13}\\ \frac{\mathbf{A}_{21}}{\varepsilon} & \frac{\mathbf{A}_{22}}{\varepsilon} & \frac{\mathbf{A}_{23}}{\varepsilon}\\ \frac{\mathbf{A}_{31}}{\mu} & \frac{\mathbf{A}_{32}}{\mu} & \frac{\mathbf{A}_{33}}{\mu}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{z}}_1\\ \hat{\mathbf{z}}_2\\ \hat{\mathbf{z}}_3\end{bmatrix} + \begin{bmatrix}\mathbf{B}_1\\ \frac{\mathbf{B}_2}{\varepsilon}\\ \frac{\mathbf{B}_3}{\mu}\end{bmatrix}\mathbf{u} + \begin{bmatrix}\mathbf{L}_1\\ \frac{\mathbf{L}_2}{\varepsilon}\\ \frac{\mathbf{L}_3}{\mu}\end{bmatrix}(\mathbf{y}-\hat{\mathbf{y}}),$$
$$= \mathbf{A}\hat{\mathbf{z}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y}-\hat{\mathbf{y}}) = (\mathbf{A}-\mathbf{L}\mathbf{C})\hat{\mathbf{z}} + \mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{y}, \quad (35)$$
$$\hat{\mathbf{y}} = \begin{bmatrix}\mathbf{C}_1 & \mathbf{C}_2 & \mathbf{C}_3\end{bmatrix}\begin{bmatrix}\hat{\mathbf{z}}_1^T & \hat{\mathbf{z}}_2^T & \hat{\mathbf{z}}_3^T\end{bmatrix}^T = \mathbf{C}\hat{\mathbf{z}}, \quad (36)$$

where $\hat{\mathbf{z}}_1$, $\hat{\mathbf{z}}_2$, and $\hat{\mathbf{z}}_3$ are estimates of $\mathbf{z}_1$, $\mathbf{z}_2$, and $\mathbf{z}_3$ respectively. As transpose of $(\mathbf{A}^T-\mathbf{C}^T\mathbf{L}^T)$ produces the dual form of $(\mathbf{A}-\mathbf{B}\mathbf{K})$, the same technique [14], [15] can be used to get $(\mathbf{A}-\mathbf{B}\mathbf{K})$ and $(\mathbf{A}-\mathbf{L}\mathbf{C})$. For that, consider system conforming to (3)-(4) as

$$\begin{bmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{x}}_2\\ \dot{\mathbf{x}}_3\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{11}^T & \frac{\mathbf{A}_{21}^T}{\varepsilon} & \frac{\mathbf{A}_{31}^T}{\mu}\\ \mathbf{A}_{12}^T & \frac{\mathbf{A}_{22}^T}{\varepsilon} & \frac{\mathbf{A}_{32}^T}{\mu}\\ \mathbf{A}_{13}^T & \frac{\mathbf{A}_{23}^T}{\varepsilon} & \frac{\mathbf{A}_{33}^T}{\mu}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix} + \begin{bmatrix}\mathbf{C}_1^T\\ \mathbf{C}_2^T\\ \mathbf{C}_3^T\end{bmatrix}\nu, \quad (37)$$

where $\mathbf{x}_1 \in \Re^{n_1}$, $\mathbf{x}_2 \in \Re^{n_2}$ and $\mathbf{x}_3 \in \Re^{n_3}$. Introducing

$$\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\\ \mathbf{q}_3\end{bmatrix} = \begin{bmatrix}\mathbf{I}_{n1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{\mathbf{I}_{n2}}{\varepsilon} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \frac{\mathbf{I}_{n3}}{\mu}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix} = \mathbf{T}_1^T\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix}, \quad (38)$$

system (37) is changed to standard singular perturbed form

$$\begin{bmatrix}\dot{\mathbf{q}}_1\\ \dot{\mathbf{q}}_2\\ \dot{\mathbf{q}}_3\end{bmatrix} = \begin{bmatrix}\mathbf{A}_{11}^T & \mathbf{A}_{21}^T & \mathbf{A}_{31}^T\\ \frac{\mathbf{A}_{12}^T}{\varepsilon} & \frac{\mathbf{A}_{22}^T}{\varepsilon} & \frac{\mathbf{A}_{32}^T}{\varepsilon}\\ \frac{\mathbf{A}_{13}^T}{\mu} & \frac{\mathbf{A}_{23}^T}{\mu} & \frac{\mathbf{A}_{33}^T}{\mu}\end{bmatrix}\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\\ \mathbf{q}_3\end{bmatrix} + \begin{bmatrix}\mathbf{C}_1^T\\ \frac{\mathbf{C}_2^T}{\varepsilon}\\ \frac{\mathbf{C}_3^T}{\mu}\end{bmatrix}\nu. \quad (39)$$

Later, (39) is structured to a lower triangular form

$$\begin{bmatrix}\dot{\mathbf{q}}_s\\ \dot{\mathbf{q}}_{fs}\\ \dot{\mathbf{q}}_3\end{bmatrix} = \begin{bmatrix}\mathbf{A}_s^T & \mathbf{0} & \mathbf{0}\\ \frac{\bar{\mathbf{A}}_{12}^T}{\varepsilon} & \frac{\mathbf{A}_{f1}^T}{\varepsilon} & \mathbf{0}\\ \frac{\mathbf{A}_{13}^T}{\mu} & \frac{\bar{\mathbf{A}}_{23}^T}{\mu} & \frac{\mathbf{A}_{f2}^T}{\mu}\end{bmatrix}\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{fs}\\ \mathbf{q}_3\end{bmatrix} + \begin{bmatrix}\mathbf{C}_s^T\\ \frac{\bar{\mathbf{C}}_{f1}^T}{\varepsilon}\\ \frac{\mathbf{C}_3^T}{\mu}\end{bmatrix}\nu, \quad (40)$$

by applying state transformation

$$\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{fs}\\ \mathbf{q}_3\end{bmatrix} = \begin{bmatrix}\mathbf{I}_{n1} & \varepsilon\mathbf{P}_{21}^T & \mu\mathbf{P}_{31}^T\\ \mathbf{0} & \mathbf{I}_{n2} & \mu\mathbf{P}_{32}^T\\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n3}\end{bmatrix}\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\\ \mathbf{q}_3\end{bmatrix} = \mathbf{T}_2^T\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\\ \mathbf{q}_3\end{bmatrix}, \quad (41)$$

to (39). Herein, $\mathbf{P}_{21}^T$ and $\mathbf{P}_{31}^T$ are reckoned by solving

$$\left.\begin{aligned}-\varepsilon\mathbf{A}_s^T\mathbf{P}_{21}^T + \mathbf{A}_{21}^T + \mathbf{P}_{21}^T\mathbf{A}_{22}^T + \mathbf{P}_{31}^T\mathbf{A}_{23}^T &= 0,\\ -\mu\mathbf{A}_s^T\mathbf{P}_{31}^T + \mathbf{A}_{31}^T + \mathbf{P}_{21}^T\mathbf{A}_{32}^T + \mathbf{P}_{31}^T\mathbf{A}_{33}^T &= 0,\end{aligned}\right\} \quad (42)$$

iteratively, in the same way as that of (9). After that, one can gain $\mathbf{A}_s^T = \mathbf{A}_{11}^T + \mathbf{P}_{21}^T\mathbf{A}_{12}^T + \mathbf{P}_{31}^T\mathbf{A}_{13}^T$, $\mathbf{A}_{f22}^T = -\varepsilon\mathbf{A}_{12}^T\mathbf{P}_{21}^T + \mathbf{A}_{22}^T$, $\mathbf{A}_{f32}^T = -\mu\mathbf{A}_{12}^T\mathbf{P}_{31}^T + \mathbf{A}_{32}^T$, $\bar{\mathbf{A}}_{23}^T = -\varepsilon\mathbf{A}_{13}^T\mathbf{P}_{21}^T + \mathbf{A}_{23}^T$, $\mathbf{A}_{f33}^T = -\mu\mathbf{A}_{13}^T\mathbf{P}_{31}^T + \mathbf{A}_{33}^T$ and $\mathbf{C}_s^T = \mathbf{C}_1^T + \mathbf{P}_{21}^T\mathbf{C}_2^T + \mathbf{P}_{31}^T\mathbf{C}_3^T$. Again, $\mathbf{P}_{32}^T$ is determined using

$$-\mu(\mathbf{A}_{f22}^T + \varepsilon\mathbf{P}_{32}^T\bar{\mathbf{A}}_{23}^T)\mathbf{P}_{32}^T + (\mathbf{A}_{f32}^T + \mathbf{P}_{32}^T\mathbf{A}_{f33}^T) = 0, \quad (43)$$

with Assumption 2 for a small value of μ. Subsequently, one can have $\bar{\mathbf{A}}_{12}^T = \mathbf{A}_{12}^T + \varepsilon\mathbf{P}_{32}^T\mathbf{A}_{13}^T$, $\mathbf{A}_{f1}^T = \mathbf{A}_{f22}^T + \varepsilon\mathbf{P}_{32}^T\bar{\mathbf{A}}_{23}^T$, $\bar{\mathbf{A}}_{f2}^T = -\mu\bar{\mathbf{A}}_{23}^T\mathbf{P}_{32}^T + \mathbf{A}_{f33}^T$ and $\bar{\mathbf{C}}_{f1}^T = \varepsilon\mathbf{P}_{32}^T\mathbf{C}_3^T + \mathbf{C}_2^T$. Since, $\lambda(\mathbf{A}) = \lambda(\mathbf{A}^T)$, the system (40) also obeys the eigenvalue clustering characteristics of the system (7). For three-stage observer design, applying

$$\nu = -\mathbf{L}_s^T\mathbf{q}_s + \nu_{f1}, \quad (44)$$

to the system (40), leads to

$$\begin{bmatrix}\dot{\mathbf{q}}_s\\ \dot{\mathbf{q}}_{fs}\\ \dot{\mathbf{q}}_3\end{bmatrix} = \begin{bmatrix}\mathbf{A}_s^T - \mathbf{C}_s^T\mathbf{L}_s^T & \mathbf{0} & \mathbf{0}\\ \frac{\bar{\mathbf{A}}_{12}^T - \bar{\mathbf{C}}_{f1}^T\mathbf{L}_s^T}{\varepsilon} & \frac{\mathbf{A}_{f1}^T}{\varepsilon} & \mathbf{0}\\ \frac{\mathbf{A}_{13}^T - \mathbf{C}_3^T\mathbf{L}_s^T}{\mu} & \frac{\bar{\mathbf{A}}_{23}^T}{\mu} & \frac{\mathbf{A}_{f2}^T}{\mu}\end{bmatrix}\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{fs}\\ \mathbf{q}_3\end{bmatrix} + \begin{bmatrix}\mathbf{C}_s^T\\ \frac{\bar{\mathbf{C}}_{f1}^T}{\varepsilon}\\ \frac{\mathbf{C}_3^T}{\mu}\end{bmatrix}\nu_{f1}, \quad (45)$$

so that $\lambda(\mathbf{A}_s^T - \mathbf{C}_s^T\mathbf{L}_s^T) = \lambda_s^{desired}$. Next, the system (45) is restructured as

$$\begin{bmatrix}\dot{\mathbf{q}}_s\\ \dot{\mathbf{q}}_{f1}\\ \dot{\mathbf{q}}_3\end{bmatrix} = \begin{bmatrix}\mathbf{A}_s^T - \mathbf{C}_s^T\mathbf{L}_s^T & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \frac{\mathbf{A}_{f1}^T}{\varepsilon} & \mathbf{0}\\ \frac{\bar{\mathbf{A}}_{13}^T}{\mu} & \frac{\bar{\mathbf{A}}_{23}^T}{\mu} & \frac{\mathbf{A}_{f2}^T}{\mu}\end{bmatrix}\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{f1}\\ \mathbf{q}_3\end{bmatrix} + \begin{bmatrix}\mathbf{C}_s^T\\ \frac{\mathbf{C}_{f1}^T}{\varepsilon}\\ \frac{\mathbf{C}_3^T}{\mu}\end{bmatrix}\nu_{f1}, \quad (46)$$

by applying state variable transformation

$$\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{f1}\\ \mathbf{q}_3\end{bmatrix} = \begin{bmatrix}\mathbf{I}_{n1} & \mathbf{0} & \mathbf{0}\\ \mathbf{P}_{12}^T & \mathbf{I}_{n2} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n3}\end{bmatrix}\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{fs}\\ \mathbf{q}_3\end{bmatrix} = \mathbf{T}_3^T\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{fs}\\ \mathbf{q}_3\end{bmatrix}, \quad (47)$$

to the system (45), in which $\bar{\mathbf{A}}_{13}^T = \mathbf{A}_{13}^T - \mathbf{C}_3^T\mathbf{L}_s^T - \bar{\mathbf{A}}_{23}^T\mathbf{P}_{12}^T$, $\mathbf{C}_{f1}^T = \bar{\mathbf{C}}_{f1}^T + \varepsilon\mathbf{P}_{12}^T\bar{\mathbf{C}}_s^T$ and

$$\varepsilon\mathbf{P}_{12}^T(\mathbf{A}_s^T - \mathbf{C}_s^T\mathbf{L}_s^T) + \bar{\mathbf{A}}_{12}^T - \bar{\mathbf{C}}_{f1}^T\mathbf{L}_s^T - \mathbf{A}_{f1}^T\mathbf{P}_{12}^T = 0. \quad (48)$$

Just like (18), (48) can be solved to get $\mathbf{P}_{12}^T$ with:

Assumption 8: $\lambda(\mathbf{A}_s^T - \mathbf{C}_s^T\mathbf{L}_s^T) \neq \lambda\left(\frac{\mathbf{A}_{f1}^T}{\varepsilon}\right)$.

Thereupon, in the second stage, passing input

$$\nu_{f1} = -\mathbf{L}_{f1}^T\mathbf{q}_{f1} + \nu_{f2}, \quad (49)$$

to the system (46) provides

$$\begin{bmatrix}\dot{\mathbf{q}}_s\\ \dot{\mathbf{q}}_{f1}\\ \dot{\mathbf{q}}_3\end{bmatrix} = \begin{bmatrix}\mathbf{A}_s^T - \mathbf{C}_s^T\mathbf{L}_s^T & -\mathbf{C}_s^T\mathbf{L}_{f1}^T & \mathbf{0}\\ \mathbf{0} & \frac{\mathbf{A}_{f1}^T - \mathbf{C}_{f1}^T\mathbf{L}_{f1}^T}{\varepsilon} & \mathbf{0}\\ \frac{\bar{\mathbf{A}}_{13}^T}{\mu} & \frac{\bar{\mathbf{A}}_{23}^T - \mathbf{C}_3^T\mathbf{L}_{f1}^T}{\mu} & \frac{\mathbf{A}_{f2}^T}{\mu}\end{bmatrix}\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{f1}\\ \mathbf{q}_3\end{bmatrix} + \begin{bmatrix}\mathbf{C}_s^T\\ \frac{\mathbf{C}_{f1}^T}{\varepsilon}\\ \frac{\mathbf{C}_3^T}{\mu}\end{bmatrix}\nu_{f2}, \quad (50)$$

to place $\lambda\left(\frac{\mathbf{A}_{f1}^T - \mathbf{C}_{f1}^T\mathbf{L}_{f1}^T}{\varepsilon}\right) = \lambda_{f1}^{desired}$. Controllability of $\left(\frac{\mathbf{A}_{f1}^T}{\varepsilon}, \frac{\mathbf{C}_{f1}^T}{\varepsilon}\right)$ is preserved due to Assumption 1. Further,

fastest states uncoupled from slow and fast states

$$\begin{bmatrix}\dot{\mathbf{q}}_s\\ \dot{\mathbf{q}}_{f1}\\ \dot{\mathbf{q}}_{f2}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_s^T-\mathbf{C}_s^T\mathbf{L}_s^T & -\mathbf{C}_s^T\mathbf{L}_{f1}^T & \mathbf{0}\\ \mathbf{0} & \frac{\mathbf{A}_{f1}^T-\mathbf{C}_{f1}^T\mathbf{L}_{f1}^T}{\varepsilon} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \frac{\mathbf{A}_{f2}^T}{\mu}\end{bmatrix}\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{f1}\\ \mathbf{q}_{f2}\end{bmatrix}+\begin{bmatrix}\mathbf{C}_s^T\\ \frac{\mathbf{C}_{f1}^T}{\varepsilon}\\ \frac{\mathbf{C}_{f2}^T}{\mu}\end{bmatrix}\nu_{f2}, \tag{51}$$

by applying linear state transformation

$$\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{f1}\\ \mathbf{q}_{f2}\end{bmatrix}=\begin{bmatrix}\mathbf{I}_{n1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{n2} & \mathbf{0}\\ \mathbf{P}_{13}^T & \mathbf{P}_{23}^T & \mathbf{I}_{n3}\end{bmatrix}\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{f1}\\ \mathbf{q}_3\end{bmatrix}=\mathbf{T}_4^T\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{f1}\\ \mathbf{q}_3\end{bmatrix}, \tag{52}$$

to (50), where $\mathbf{C}_{f2}^T=\mu\mathbf{P}_{13}^T\mathbf{C}_s^T+\frac{\mu}{\varepsilon}\mathbf{P}_{23}^T\mathbf{C}_{f1}^T+\mathbf{C}_3^T$ and $\mathbf{P}_{13}^T$ and $\mathbf{P}_{23}^T$ are estimated by computing

$$\mu\mathbf{P}_{13}^T(\mathbf{A}_s^T-\mathbf{C}_s^T\mathbf{L}_s^T)+\bar{\mathbf{A}}_{13}^T-\mathbf{A}_{f2}^T\mathbf{P}_{13}^T=0, \tag{53}$$

$$-\mu\mathbf{P}_{13}^T\mathbf{C}_s^T\mathbf{L}_{f1}^T+\frac{\mu}{\varepsilon}\mathbf{P}_{23}^T(\mathbf{A}_{f1}^T-\mathbf{C}_{f1}^T\mathbf{L}_{f1}^T)+\bar{\mathbf{A}}_{23}-\mathbf{C}_3^T\mathbf{L}_{f1}^T-\bar{\mathbf{A}}_{f2}^T\mathbf{P}_{23}^T=0, \tag{54}$$

with Assumptions 9 and 10 for (53) and (54) respectively.

Assumption 9: $\lambda(\mathbf{A}_s^T-\mathbf{C}_s^T\mathbf{L}_s^T)\neq\lambda\left(\frac{\mathbf{A}_{f2}^T}{\mu}\right)$.

Assumption 10: $\lambda\left(\frac{\mathbf{A}_{f1}^T-\mathbf{C}_{f1}^T\mathbf{L}_{f1}^T}{\varepsilon}\right)\neq\lambda\left(\frac{\mathbf{A}_{f2}^T}{\mu}\right)$.

Finally,

$$\nu_{f2}=-\mathbf{L}_{f2}^T\mathbf{q}_{f2}, \tag{55}$$

is applied to the system (51), in the third stage, as

$$\begin{bmatrix}\dot{\mathbf{q}}_s\\ \dot{\mathbf{q}}_{f1}\\ \dot{\mathbf{q}}_{f2}\end{bmatrix}=\begin{bmatrix}\mathbf{A}_s^T-\mathbf{C}_s^T\mathbf{L}_s^T & -\mathbf{C}_s^T\mathbf{L}_{f1}^T & -\mathbf{C}_s^T\mathbf{L}_{f2}^T\\ \mathbf{0} & \frac{\mathbf{A}_{f1}^T-\mathbf{C}_{f1}^T\mathbf{L}_{f1}^T}{\varepsilon} & -\mathbf{C}_{f1}^T\mathbf{L}_{f2}^T\\ \mathbf{0} & \mathbf{0} & \frac{\mathbf{A}_{f2}^T-\mathbf{C}_{f2}^T\mathbf{L}_{f2}^T}{\mu}\end{bmatrix}\begin{bmatrix}\mathbf{q}_s\\ \mathbf{q}_{f1}\\ \mathbf{q}_{f2}\end{bmatrix}, \tag{56}$$

with aim to assign $\lambda\left(\frac{\mathbf{A}_{f2}^T-\mathbf{C}_{f2}^T\mathbf{L}_{f2}^T}{\mu}\right)=\lambda_{f2}^{desired}$, assuming the controllability of $\left(\frac{\mathbf{A}_{f2}^T}{\mu},\frac{\mathbf{C}_{f2}^T}{\mu}\right)$ is satisfied by Assumption 1. The overall observer gain is given by

$$\nu=\mathbf{L}^T\mathbf{x}=\begin{bmatrix}\mathbf{L}_1^T & \frac{1}{\varepsilon}\mathbf{L}_2^T & \frac{1}{\mu}\mathbf{L}_3^T\end{bmatrix}\begin{bmatrix}\mathbf{x}_1^T & \mathbf{x}_2^T & \mathbf{x}_3^T\end{bmatrix}^T, \tag{57}$$

where $\mathbf{L}_1^T=\mathbf{L}_s^T+\mathbf{L}_f^T\mathbf{P}_{13}^T+(\mathbf{L}_{f2}^T\mathbf{P}_{23}^T+\mathbf{L}_{f1}^T)\mathbf{P}_{12}^T$, $\mathbf{L}_2^T=\varepsilon\mathbf{L}_1^T\mathbf{P}_{21}^T+\mathbf{L}_{f2}^T\mathbf{P}_{23}^T+\mathbf{L}_{f1}^T$, and $\mathbf{L}_3^T=\mu\mathbf{L}_1^T\mathbf{P}_{31}^T+\mu(\mathbf{L}_{f2}^T\mathbf{P}_{23}^T+\mathbf{L}_{f1}^T)\mathbf{P}_{32}^T+\mathbf{L}_{f2}^T$. Once again, applying

$$\begin{bmatrix}\mathbf{x}_s^T & \mathbf{x}_{f1}^T & \mathbf{x}_{f2}^T\end{bmatrix}^T=\mathbf{T}_1^{-T}\begin{bmatrix}\mathbf{q}_s^T & \mathbf{q}_{f1}^T & \mathbf{q}_{f2}^T\end{bmatrix}^T, \tag{58}$$

to the system (56) results

$$\begin{bmatrix}\dot{\mathbf{x}}_s\\ \dot{\mathbf{x}}_{f1}\\ \dot{\mathbf{x}}_{f2}\end{bmatrix}=\begin{bmatrix}(\mathbf{A}_s-\mathbf{L}_s\mathbf{C}_s)^T & \frac{-(\mathbf{L}_{f1}\mathbf{C}_s)^T}{\varepsilon} & \frac{-(\mathbf{L}_{f2}\mathbf{C}_s)^T}{\mu}\\ \mathbf{0} & \frac{(\mathbf{A}_{f1}-\mathbf{L}_{f1}\mathbf{C}_{f1})^T}{\varepsilon} & \frac{-(\mathbf{L}_{f2}\mathbf{C}_{f1})^T}{\mu}\\ \mathbf{0} & \mathbf{0} & \frac{(\mathbf{A}_{f2}-\mathbf{L}_{f2}\mathbf{C}_{f2})^T}{\mu}\end{bmatrix}\begin{bmatrix}\mathbf{x}_s\\ \mathbf{x}_{f1}\\ \mathbf{x}_{f2}\end{bmatrix}. \tag{59}$$

The hypothetical systems (37) and (59) are related as

$$\left.\begin{aligned}\begin{bmatrix}\mathbf{x}_s^T & \mathbf{x}_{f1}^T & \mathbf{x}_{f2}^T\end{bmatrix}^T&=\mathbf{T}^T\begin{bmatrix}\mathbf{x}_1^T & \mathbf{x}_2^T & \mathbf{x}_3^T\end{bmatrix}^T,\\ \begin{bmatrix}\mathbf{x}_1^T & \mathbf{x}_2^T & \mathbf{x}_3^T\end{bmatrix}^T&=\mathbf{T}^{-T}\begin{bmatrix}\mathbf{x}_s^T & \mathbf{x}_{f1}^T & \mathbf{x}_{f2}^T\end{bmatrix}^T,\end{aligned}\right\} \tag{60}$$

where $\mathbf{T}^T=\mathbf{T}_1^{-T}\mathbf{T}_4^T\mathbf{T}_3^T\mathbf{T}_2^T\mathbf{T}_1^T$. Using (59) and the duality principle [11], [14], observer can be configured

$$\begin{bmatrix}\dot{\hat{\mathbf{z}}}_s\\ \dot{\hat{\mathbf{z}}}_{f1}\\ \dot{\hat{\mathbf{z}}}_{f2}\end{bmatrix}=\begin{bmatrix}(\mathbf{A}_s-\mathbf{L}_s\mathbf{C}_s) & \mathbf{0} & \mathbf{0}\\ \frac{-(\mathbf{L}_{f1}\mathbf{C}_s)}{\varepsilon} & \frac{(\mathbf{A}_{f1}-\mathbf{L}_{f1}\mathbf{C}_{f1})}{\varepsilon} & \mathbf{0}\\ \frac{-(\mathbf{L}_{f2}\mathbf{C}_s)}{\mu} & \frac{-(\mathbf{L}_{f2}\mathbf{C}_{f1})}{\mu} & \frac{(\mathbf{A}_{f2}-\mathbf{L}_{f2}\mathbf{C}_{f2})}{\mu}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{z}}_s\\ \hat{\mathbf{z}}_{f1}\\ \hat{\mathbf{z}}_{f2}\end{bmatrix}+\begin{bmatrix}\mathbf{B}_s\\ \frac{\mathbf{B}_{f1}}{\varepsilon}\\ \frac{\mathbf{B}_{f2}}{\mu}\end{bmatrix}\mathbf{u}+\begin{bmatrix}\mathbf{L}_s\\ \frac{\mathbf{L}_{f1}}{\varepsilon}\\ \frac{\mathbf{L}_{f2}}{\mu}\end{bmatrix}\mathbf{y}. \tag{61}$$

Employing (60) to system (35)-(36) gives

$$\begin{bmatrix}\hat{\mathbf{z}}_1^T & \hat{\mathbf{z}}_2^T & \hat{\mathbf{z}}_3^T\end{bmatrix}^T=\mathbf{T}\begin{bmatrix}\hat{\mathbf{z}}_s^T & \hat{\mathbf{z}}_{f1}^T & \hat{\mathbf{z}}_{f2}^T\end{bmatrix}^T, \tag{62}$$

resulting into

$$\begin{bmatrix}\dot{\hat{\mathbf{z}}}_s\\ \dot{\hat{\mathbf{z}}}_{f1}\\ \dot{\hat{\mathbf{z}}}_{f2}\end{bmatrix}=\mathbf{T}^{-1}(\mathbf{A}-\mathbf{L}\mathbf{C})\mathbf{T}\begin{bmatrix}\hat{\mathbf{z}}_s\\ \hat{\mathbf{z}}_{f1}\\ \hat{\mathbf{z}}_{f2}\end{bmatrix}+\mathbf{T}^{-1}\mathbf{B}\mathbf{u}+\mathbf{T}^{-1}\mathbf{L}\mathbf{y}. \tag{63}$$

Systems (61) and (63) are equivalent, in which case

$$\mathbf{T}^{-1}\mathbf{L}=\begin{bmatrix}\mathbf{L}_s^T & \frac{\mathbf{L}_{f1}^T}{\varepsilon} & \frac{\mathbf{L}_{f2}^T}{\mu}\end{bmatrix}^T\Longrightarrow\mathbf{L}=\mathbf{T}\begin{bmatrix}\mathbf{L}_s^T & \frac{\mathbf{L}_{f1}^T}{\varepsilon} & \frac{\mathbf{L}_{f2}^T}{\mu}\end{bmatrix}^T,$$

can be verified with $\mathbf{L}$ in (57) and $\begin{bmatrix}\mathbf{B}_s^T & \frac{\mathbf{B}_{f1}^T}{\varepsilon} & \frac{\mathbf{B}_{f2}^T}{\mu}\end{bmatrix}^T$ can be calculated using $\mathbf{T}^{-1}\mathbf{B}$.

IV. NUMERICAL EXAMPLE

Consider a 9th order system [16], having 3 slow, 2 fast and 4 fastest eigenvalues as

$$\mathbf{A}=\begin{bmatrix}-0.311 & 0 & 1.028 & 0 & 0 & 2\\ 1.01 & -0.65 & -0.084 & 0 & 0.83 & 1.70\\ -0.11 & 0 & -1.26 & 0 & -6.30 & -1.04\\ 0 & 0.10 & 0 & -13.00 & 0 & 0.48\\ 0 & -0.05 & 0.25 & 0 & -17.60 & -0.30\\ 0 & 4.50 & -3.80 & -640 & 375 & -97.50\\ 0 & -2.90 & 2.75 & 11.10 & 0 & 1.01\\ 2.50 & -5.70 & -4.00 & 0 & 0 & -0.91\\ 1.80 & -3.10 & 3.18 & 0 & 1.76 & -0.70\end{bmatrix}$$

$$\begin{bmatrix}0.10 & -6.30 & -0.40\\ 3 & 2.40 & 0.20\\ 0 & 4.10 & 0.01\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 1.12 & 0 & -0.80\\ -143 & 0.50 & 0\\ 0 & -184 & -4.20\\ 1.53 & 3.20 & -207\end{bmatrix},\mathbf{B}=\begin{bmatrix}0.8812\\ 1.1366\\ -0.6790\\ -0.1311\\ -0.2270\\ 0.9823\\ -1.0174\\ -0.8227\\ 0.7900\end{bmatrix},\mathbf{C}=\begin{bmatrix}1\\ 1\\ 1\\ 1\\ 1\\ 1\\ 1\\ 1\\ 1\end{bmatrix}^T.$$

Eigenvalues are located at (-206.40, -184.35, -142.96, -92.26, -20.38, -15.31, -1.56, -0.80, -0.30) with $\varepsilon=0.1021,\mu=0.0169$. Thus, N_1 and N_2 are is taken as 4 and 5 respectively. If $\tau=2$ s, then $\Delta=0.5$ s and $\delta=0.1$ s. Suppose the desired eigenvalues are (-180, -150, -130, -110, -18, -16, -2.6, -1.8, -0.8). Now, solutions of (9) and (10) are found and system (7) is reached, which is then discretized with 0.1 s. Later, to place $\lambda_{\delta f2}^{desired}$, the feedback gain $\mathbf{F}_{f2}$ is determined. Next, $\mathbf{M}_{23}$ is estimated to get (16) and then represented as (19) with Δ sampling

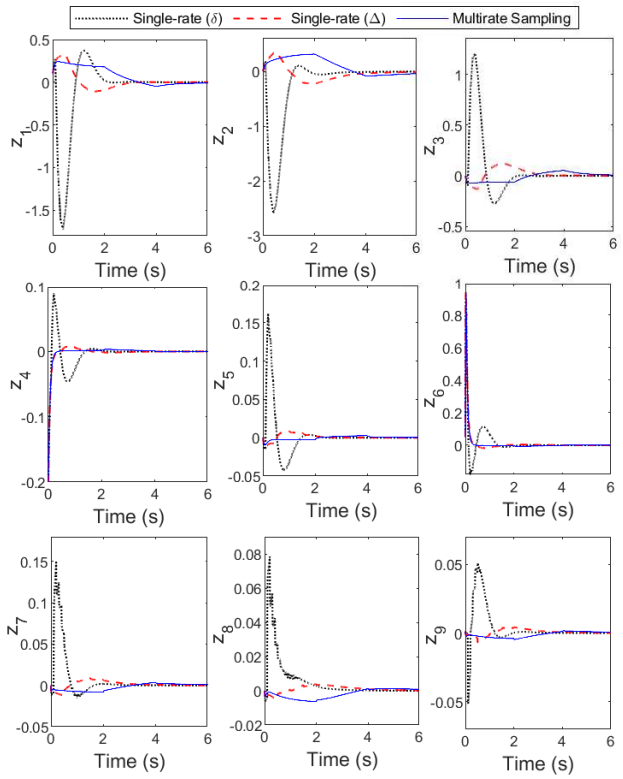


Fig. 1. Evolution of states of the original three-time-scale system.

interval using (20). The fast subsystem eigenvalues are placed at $\lambda_{\Delta f1}^{desired}$ by computing $\mathbf{F}_{f1}$. Finally, $\mathbf{M}_{12}$ and $\mathbf{M}_{13}$ are determined to get (23) and then represented into (27) with τ sampling interval using (28). The feedback gain $\mathbf{F}_s$ is computed to place placed $\lambda_{\tau s}^{desired}$. All these are given below.

$$\mathbf{F}_{f2} = \begin{bmatrix} 0.0079 & 0.0002 & -0.0000 & -0.0001 \end{bmatrix},$$
$$\mathbf{F}_{f1} = \begin{bmatrix} -0.0118 & 0.0049 \end{bmatrix}, \mathbf{F}_s = \begin{bmatrix} 0.9350 & 0.1143 & 0.9972 \end{bmatrix}.$$

Utilizing obtained feedback gains, control input (31) is formulated. System initial conditions are taken as (0.1, 0, 0, -0.2, 0, 0.05, 0, 0, 0). States $\hat{\mathbf{z}}_s^T$, $\hat{\mathbf{z}}_{f1}^T$ and $\hat{\mathbf{z}}_{f2}^T$ are estimated using the sequential observer and are sampled at τ, Δ and δ sampling intervals respectively. Desired observer eigenvalues are taken as (-540, -450, -390, -310, -54, -48, -7.8, -5.4, -2.7). As that of state feedback gain, the observer gains are determined as

$$\mathbf{L}_s^T = \begin{bmatrix} 27.0435 & -58.0586 & 37.5108 \end{bmatrix},$$
$$\mathbf{L}_{f1}^T = \begin{bmatrix} -3.1335 & -3.2063 \end{bmatrix},$$
$$\mathbf{L}_{f2}^T = \begin{bmatrix} 96.74 & -357.93 & 279.31 & -71.05 \end{bmatrix}.$$

Using all these feedback and observer gains, system is simulated. The simulation results are compared with the single-rate sampling viz. δ and Δ seconds. Fig. 1 shows the evolution of original states showing performance improvement with MRSF control.

V. CONCLUSION

This paper presents MRSF control for a three-time-scale system using the accurate division of the fastest, fast, and slow subsystems. The relation between the sampling periods of the fastest, fast, and slow subsystems is obtained. Three steps of subsystem separation and three stages of MRSF control are merged. For control implementation, the decoupled states are computed by a sequential three-stage observer. The proposed control is then applied to an illustrative example and simulations are presented. Simulations are compared with the single-rate sampling to show improvement in the system performance.

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