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Funnel-based Control for Reach-Avoid-Stay Specifications

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***Abstract*— The paper addresses the problem of controller synthesis for control-affine nonlinear systems to meet reach-avoid-stay specifications. Specifically, the goal of the research is to obtain a closed-form control law ensuring that the trajectories of the nonlinear system reach a target set while avoiding all unsafe regions and adhering to the state-space constraints. To tackle this problem, we leverage the concept of the funnel-based control approach. Given an arbitrary unsafe region, we introduce a circumvent function that guarantees the system trajectory to steer clear of that region. Subsequently, an adaptive funnel framework is proposed based on the target, followed by the construction of a closed-form controller using the established funnel function, enforcing the reach-avoid-stay specifications. To demonstrate the efficacy of the proposed funnel-based control approach, a series of simulation experiments have been carried out.**

I. Introduction

Recent years have seen a surge in interest in reach-avoid-stay (RAS) specifications for ensuring safety in autonomous systems, wherein trajectories must reach a target while avoiding unsafe sets and respecting state constraints. Synthesizing controllers for RAS specs is crucial for complex tasks [1] and robust control in safety-critical scenarios like trajectory regulation, motion planning, and obstacle avoidance.

With the onset of the usage of formal languages for specifying complex tasks, symbolic control [2], [3] has emerged as a powerful tool. Despite efforts to enhance efficiency [4], [5], these abstraction-based approaches still face challenges related to the so-called curse of dimensionality. In contrast, nonlinear approaches like barrier-based control [6] ensure safety and stability without state-space discretization. Authors in [7] and [8] have proposed implementing control Lyapunov-barrier functions to characterize reach-avoid-stay specifications. Yet reliance on optimization techniques for controller synthesis can increase computational complexity, especially for large systems.

Funnel-based control [9] offers advantages in designing closed-loop schemes meeting tracking requirements. Numerous successful applications have been reported in the literature [10], from solving tracking control problems for unknown nonlinear systems [11] to handling multi-agent systems subjected to complex task specifications [12]. Although, it effectively enforces reachability specifications [13], [14], active obstacle avoidance remains challenging. Some studies [15] have described a two-step process involving trajectory creation through path-planning algorithms and then applying funnel-based feedback control. However, separating the trajectory planning from funnels could compromise avoidance constraints under disturbances.

In this work, we demonstrated skipping the trajectory generation step by leveraging the funnel functions' ability to navigate around unsafe sets. In this context, we introduced the circumvent function to define the avoid specifications and modifying the funnel constraints. Subsequently, we formulate an approximation-free closed-form control law that ensures the system's trajectory remains within the modified funnel, thereby achieving RAS objectives. The effectiveness of the proposed approach is demonstrated through a robot navigation simulation study.

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II. Preliminaries and Problem Formulation

A. Notations

The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}_0^+$ denote the set of natural, real, positive real, and nonnegative real numbers, respectively. We use $\mathbb{R}^{n\times m}$ to denote a vector space of real matrices with n rows and m columns. To represent a column vector with n rows, we use $\mathbb{R}^n$. We represent the Euclidean norm using $\|\cdot\|$. For $a, b \in \mathbb{R}$ and $a < b$, we use (a, b) to represent open interval in $\mathbb{R}$. For $a, b \in \mathbb{N}$ and $a \leq b$, we use $[a; b]$ to denote close interval in $\mathbb{N}$. To denote a vector $x \in \mathbb{R}^n$ with entries $x_1, \ldots, x_n$, we use $\mathsf{col}(x_1, \ldots, x_n)$, where $x_i \in \mathbb{R}, i \in [1; n]$ denotes i-th element of vector $x \in \mathbb{R}^n$. A diagonal matrix in $\mathbb{R}^{n\times n}$ with diagonal entries $d_1, \ldots, d_n$ is denoted by $\mathsf{diag}(d_1, \ldots, d_n)$. Given $N \in \mathbb{N}$ sets $\mathbf{X}_i$, $i \in [1; N]$, the Cartesian product of the sets is given by $\mathbf{X} = \prod_{i\in[1;N]} \mathbf{X}_i := \{(x_1, \ldots, x_N) | x_i \in \mathbf{X}_i, i \in [1; N]\}$. Consider a set $\mathbf{X}_a \subset \mathbb{R}^n$, its projection on ith dimension, where $i \in [1; n]$, is given by an interval $[\underline{\mathbf{X}}_{ai}, \overline{\mathbf{X}}_{ai}] \subset \mathbb{R}$, where $\underline{\mathbf{X}}_{ai} := \min\{x_i \in \mathbb{R} \mid [x_1 \ldots, x_n] \in \mathbf{X}_a\}$ and $\overline{\mathbf{X}}_{ai} := \max\{x_i \in \mathbb{R} \mid [x_1, \ldots, x_n] \in \mathbf{X}_a\}$. We further define the hyper-rectangle $[\![\mathbf{X}_a]\!] = \prod_{i=[1;n]} [\underline{\mathbf{X}}_{ai}, \overline{\mathbf{X}}_{ai}]$. We denote the empty set by $\emptyset$. The space of bounded continuous functions is denoted by $\mathcal{C}$. Given a compact set $\mathbf{X}$, $int(\mathbf{X})$ represents the interior of the set and $\partial\mathbf{X} = \mathbf{X} \setminus int(\mathbf{X})$ represents the boundary of $\mathbf{X}$. $\overline{\max}$ and $\overline{\min}$ are smooth approximations of the non-smooth $\max$ and $\min$ functions, defined as, $\overline{\max}(a, b) \approx \frac{1}{\nu}\ln(\mathsf{e}^{\nu a} + \mathsf{e}^{\nu b})$ and $\overline{\min}(a, b) \approx -\frac{1}{\nu}\ln(\mathsf{e}^{-\nu a} + \mathsf{e}^{-\nu b})$, respectively. The sign function $\mathsf{sign}(x) := -1$ if $x < 0$, and $\mathsf{sign}(x) := 1$ if $x \geq 0$.

B. System Definition

Consider the following control-affine nonlinear system:

$$\mathcal{S} : \dot{x} = f(x) + g(x)u, \tag{1}$$

where $x(t) = \mathsf{col}(x_1(t), \ldots, x_n(t)) \in \mathbf{X} \subset \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and control input vectors, respectively. The state space of the system is defined by the closed and connected set $\mathbf{X}$. The functions $f : \mathbf{X} \to \mathbb{R}^n$ and $g : \mathbf{X} \to \mathbb{R}^{n\times m}$ satisfy Assumption 1.

Assumption 1: f and g are locally Lipschitz, and $g(x)g^\top(x)$ is positive definite for all $x \in \mathbb{R}^n$.

C. Problem Formulation

Let the compact set $\mathbf{T} \subset \mathbf{X}$ be the target set, the set $\mathbf{U} = \bigcup_{j\in[1;n_u]} \mathcal{U}^j$ be an unsafe region containing $n_u \in \mathbb{N}$ compact unsafe sets $\mathcal{U}^j \subset \mathbf{X}$. Note that, in general, the unsafe region $\mathbf{U}$, although necessarily compact, can be nonconvex and even disconnected.

Problem 2.1: Given a control-affine system $\mathcal{S}$ in (1) with Assumption 1, target set $\mathbf{T} \subset \mathbf{X}$, and unsafe region $\mathbf{U}$, as defined above, design a closed-form controller to ensure the satisfaction of the reach-avoid-stay specification, i.e., for a given initial position $x(0) \in \mathbf{X} \setminus \mathbf{U}$, there exists $t \in \mathbb{R}_0^+$, such that, $x(t) \in \mathbf{T}$ and for all $s \in [0, t] : x(s) \in \mathbf{X} \setminus \mathbf{U}$.

Remark 2.2: If $\mathbf{X}$ is of any arbitrary shape, we redefine the state space as the hyper-rectangle $\hat{\mathbf{X}} := [\![\mathbf{X}]\!] = \prod_{i\in[1;n]}[\underline{\mathbf{X}}_i, \overline{\mathbf{X}}_i]$ and expand the unsafe region $\hat{\mathbf{U}} = \mathbf{U} \cup ([\![\mathbf{X}]\!] \setminus \mathbf{X})$. Here, $[\underline{\mathbf{X}}_i, \overline{\mathbf{X}}_i]$ represent the projection of set $\mathbf{X}$ on the ith dimension. Note that, adding $[\![\mathbf{X}]\!] \setminus \mathbf{X}$ to the unsafe set $\mathbf{U}$ and following Algorithm 1 in Section V, enforces stay specifications for an arbitrary state-space $\mathbf{X}$.

III. Controller for Reachability Specification

In this section, we formulate a funnel-based control strategy [9] aimed at guaranteeing that the system's trajectory starting from $x(0) \in \mathbf{X}$ reaches a target set $\mathbf{T} \subset \mathbf{X}$. We first define the funnel constraints over the trajectory as follows:

$$\underbrace{-\underline{c}_i\rho_i(t) + \eta_i}_{\rho_{i,L}(t)} < x_i(t) < \underbrace{\overline{c}_i\rho_i(t) + \eta_i}_{\rho_{i,U}(t)}, \forall i \in [1; n], \quad (2)$$

where $\eta = \mathsf{col}(\eta_1, \ldots, \eta_n) \in int(\mathbf{T})$, $\underline{c}_i = \eta_i - \underline{\mathbf{X}}_i$ and $\overline{c}_i = \overline{\mathbf{X}}_i - \eta_i$. $\rho_i(t)$ is the continuously differentiable, positive and non-increasing funnel function defined as:

$$\rho_i(t) = (\rho_{i,0} - \rho_{i,\infty})e^{-l_i t} + \rho_{i,\infty} \quad (3)$$

with $\rho_{i,0} = 1$, $\rho_{i,\infty} \in \left(0, \min\left(\rho_{i,0}, \frac{|\mathbf{T}_i - \eta_i|}{\max\{\underline{c}_i, \overline{c}_i\}}\right)\right)$ and $l_i \in \mathbb{R}_0^+$ governs the lower bound of convergence rate.

$\rho_{i,0}$, $\rho_{i,\infty}$, $\underline{c}_i$, and $\overline{c}_i$ ensures $x_i(0) \in [\underline{\mathbf{X}}_i, \overline{\mathbf{X}}_i], \forall i \in [1; n]$ and $t \to \infty, x(t) \in \prod_{i\in[1;n]}(\eta_i + [-\underline{c}_i\rho_{i,\infty}, \overline{c}_i\rho_{i,\infty}]) \subset \mathbf{T}$. An example of a funnel designed for enforcing reach specification is shown in Figure 1 (a).

To design a controller enforcing condition (2), we define the normalized error $e(x, t) = \mathsf{col}(e_1(x_1, t), \ldots e_n(x_n, t))$, transformed error $\varepsilon = \mathsf{col}(\varepsilon_1, \ldots, \varepsilon_n)$, and the diagonal matrix $\xi(x, t) = \mathsf{diag}(\xi_1(x, t), \ldots, \xi_n(x, t))$ as:

$$e_i(x_i, t) = \frac{x_i(t) - \frac{1}{2}(\rho_{i,U}(t) + \rho_{i,L}(t))}{\frac{1}{2}(\rho_{i,U}(t) - \rho_{i,L}(t))}, \forall i \in [1; n]. \quad (4)$$

$$\varepsilon_i(x, t) = y(e_i(x, t)) = \ln\left(\frac{1 + e_i(x, t)}{1 - e_i(x, t)}\right), \forall i \in [1; n]. \quad (5)$$

$$\xi_i(x, t) = \frac{4}{(\rho_{i,U} - \rho_{i,L})(1 - e_i(x, t)^2)}, \forall i \in [1; n]. \quad (6)$$

Note, if the transformed error $\varepsilon(x, t)$ is bounded, then the normalized error $e(x, t)$ is within the constrained region $\mathbb{D} := (-1, 1)$ and the state $x(t)$ adheres to (2).

Theorem 3.1: Consider the control-affine system $\mathcal{S}$ given in (1) with Assumptions 1. Given a target set $\mathbf{T}$, the funnel constraints $\rho_{i,U}(t)$ and $\rho_{i,L}(t)$ (2), the control strategy

$$u(x, t) = -g(x)^\top(g(x)g(x)^\top)^{-1} \left(k\xi(x, t)\varepsilon(x, t) - \frac{1}{2}\dot{\rho}_d(t)e(x, t)\right) \quad (7)$$

will drive the state $x(t)$ to the target set $\mathbf{T}$ in finite time, i.e., $\exists t \in \mathbb{R}_0^+ : x(t) \in \mathbf{T}$. Here, $k \in \mathbb{R}^+$ is a constant, $\rho_d := \mathsf{diag}(\rho_{1,d}, \ldots, \rho_{n,d})$, with $\rho_{i,d} = \rho_{i,U} - \rho_{i,L}$.

Proof: The proof follows on similar grounds as that of Theorem 4.2 and is omitted here due to space constraints. ■

Thus, given a system $\mathcal{S}$ in (1), a target set $\mathbf{T}$ in the state space $\mathbf{X}$, we can define a funnel and the closed-form well-defined control law (7) that will guide the system trajectory to the target, enforcing reachability specifications.

IV. Extension to Reach-Avoid-Stay Specification

In this section, we begin by exploring the integration of avoidance of unsafe regions within the funnel-based control framework. Subsequently, we present an adaptive funnel design strategy that enables the successful accomplishment of reach-avoid-stay tasks.

A. Design of Circumvent Function

Consider an unsafe region $\mathbf{U}$ with n_u compact sets $\mathcal{U}^j$, for $j \in [1; n_u]$. Note that although we assume $\mathcal{U}^j$ to be convex, $\mathbf{U}$ can be non-convex. Section V will further elaborate on this.

First, given an initial state $x(0) \in \mathbf{X} \setminus \mathbf{U}$, we obtain the time range $[\underline{t}^j, \overline{t}^j]$ over which the system trajectory $x(t)$, on application of the control law $u(x, t)$ (7) intersects with the jth unsafe set $\mathcal{U}^j$, and is given by, $\underline{t}^j = \inf\{t \in \mathbb{R}^+ : x(t) \cap \mathcal{U}^j \neq \emptyset\}$ and $\overline{t}^j = \sup\{t \in \mathbb{R}^+ : x(t) \cap \mathcal{U}^j \neq \emptyset\}$. Let the system trajectory first intersect $\mathcal{U}^{\hat{j}}$, where

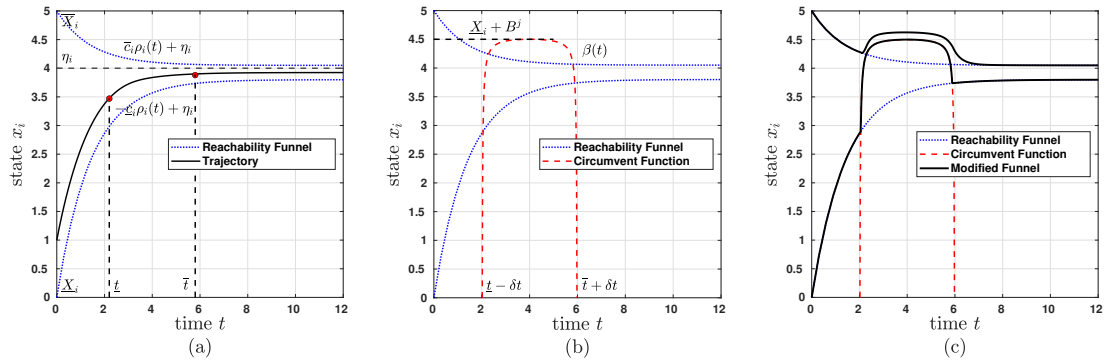
$$\hat{j} = \arg\min_{j\in[1;n_u]} \underline{t}^j. \quad (8)$$

If the system's trajectory enters the unsafe set $\mathcal{U}^{\hat{j}}$, then $\exists t \in \mathbb{R}^+$, such that $x_i(t) \cap [\underline{\mathcal{U}}_i^{\hat{j}}, \overline{\mathcal{U}}_i^{\hat{j}}] \neq \emptyset, \forall i \in [1; n]$. Hence, to satisfy the avoid specification, it is sufficient to introduce the circumvent function only in one dimension $i^{\hat{j}}$, given by

$$i^{\hat{j}} = \arg\min_{i\in[1;n]} t_i^{\hat{j}}, \quad (9)$$

where $t_i^{\hat{j}} = \inf\{t \in \mathbb{R}^+ : x_i(t) \cap [\underline{\mathcal{U}}_i^{\hat{j}}, \overline{\mathcal{U}}_i^{\hat{j}}] \neq \emptyset\}$.

We can choose to modify either the upper or lower constraint boundary of the tube, except when $\underline{\mathcal{U}}^j i = \underline{\mathbf{X}} i$ or $\overline{\mathcal{U}}^j i = \overline{\mathbf{X}} i$. In such cases, where the obstacle is analogous to a wall-shaped obstacle scenario with no space between the state space boundary and the obstacle at one end, the



(a) (b) (c)

Fig. 1: Funnel Design. (a) Reachability funnel to obtain $\underline{t}$ and $\overline{t}$. (b) Circumvent function. (c) Adaptive funnel framework.

circumvent function must be introduced in the lower and upper constraint boundaries, respectively.

The circumvent function on lower constraint for $i = i^{\hat{j}}$ is:

$$\beta_i^{\hat{j}}(t) = \begin{cases} B^{\hat{j}} \mathsf{e}^{\frac{-k^{\hat{j}}\left(t-m^{\hat{j}}\right)^2}{(r^{\hat{j}})^2-\left(t-m^{\hat{j}}\right)^2}} + \underline{\mathbf{X}}_i, & \forall t \in T_{act} \\ \underline{\mathbf{X}}_i, & \forall t \in \mathbb{R}^+ \setminus T_{act} \end{cases} \tag{10}$$

where, $B^{\hat{j}} = \overline{\mathcal{U}}_i^{\hat{j}} - \underline{\mathbf{X}}_{ai} + \delta B$, $m^{\hat{j}} := \frac{\underline{t}^{\hat{j}}+\overline{t}^{\hat{j}}}{2}$, $r^{\hat{j}} := \frac{\underline{t}^{\hat{j}}-\overline{t}^{\hat{j}}}{2} + \delta t$ and $\delta t \in \mathbb{R}^+$ is a tolerance factor. The function is active in the time range $T_{act} = [\underline{t}^{\hat{j}} - \delta t, \overline{t}^{\hat{j}} + \delta t]$ when the system trajectory avoids $\mathcal{U}^{\hat{j}}$. The δB governs how far from $\mathcal{U}^{\hat{j}}$ should the trajectory stay clear. $k^{\hat{j}} \in \mathbb{R}^+$ determines the smoothness of the circumvent function.

Similarly, the circumvent on the upper constraint is

$$\beta_i^{\hat{j}}(t) = \begin{cases} -B^{\hat{j}} \mathsf{e}^{\frac{-k^{\hat{j}}\left(t-m^{\hat{j}}\right)^2}{(r^{\hat{j}})^2-\left(t-m^{\hat{j}}\right)^2}} + \overline{\mathbf{X}}_i, & \forall t \in T_{act} \\ \overline{\mathbf{X}}_i, & \forall t \in \mathbb{R}^+ \setminus T_{act} \end{cases} \tag{11}$$

with $B^{\hat{j}} = \overline{\mathbf{X}}_i - \underline{\mathcal{U}}_i^{\hat{j}} + \delta B$ and the rest of the parameters are the same as above.

An example of the introduction of the circumvent function on the lower constraint of a funnel is shown in Figure 1 (b).

B. Adaptive Funnel Design

Given a target set $\mathbf{T}$ and obstacle $\mathcal{U}^{\hat{j}}$, choose a point $\eta \in int(\mathbf{T} \setminus \mathcal{U}^{\hat{j}})$. Now, according to (2), construct the funnel ρ_L and ρ_U to satisfy the reachability specification. As defined in the previous subsection, we characterize the obstacle using the circumvent function $\beta^{\hat{j}}(t)$, and now, we incorporate it into the funnel design. To solve Problem 2.1, we propose an adaptive framework akin to [13] and construct the following adaptive funnel constraints:

$$\text{If } \beta_i^{\hat{j}} \text{ introduced on lower constraint } \rho_{i,L}: \begin{cases} \gamma_{i,L}(t) := \overline{\max}(\rho_{i,L}(t), \beta_i^{\hat{j}}(t)), \\ \gamma_{i,U}(t) := \rho_{i,U}(t) + \alpha_i(t), \end{cases} \tag{12}$$

$$\text{If } \beta_i^{\hat{j}} \text{ introduced on upper constraint } \rho_{i,U}: \begin{cases} \gamma_{i,L}(t) := \rho_{i,L}(t) - \alpha_i(t), \\ \gamma_{i,U}(t) := \overline{\min}(\rho_{i,U}(t), \beta_i^{\hat{j}}(t)), \end{cases} \tag{13}$$

The modifications in the constraints of the funnel are captured by a continuously differentiable update function, $\alpha(t) = \mathsf{col}(\alpha_1(t), \ldots, \alpha_n(t))$. The adaptive law governing the dynamics of the update function is defined as:

$$\dot{\alpha}_i(t) = \frac{\theta_i(t)}{\psi_i(t) + \alpha_i(t)} - \kappa \alpha_i(t), \ \alpha_i(0) = 0, \tag{14}$$

where $\psi_i(t) = \rho_{i,U}(t) - \beta_i^{\hat{j}}(t) - \mu$ if $\beta_i^{\hat{j}}$ introduced on lower constraint $\rho_{i,L}$ and $\psi_i(t) = \beta_i^{\hat{j}}(t) - \rho_{i,L}(t) - \mu$ if $\beta_i^{\hat{j}}$ introduced on upper constraint $\rho_{i,U}$, with $\mu \in \mathbb{R}^+$ as a tolerance factor. $\theta_i(t)$ acts as a trigger, activating the first part of the update function only when reach-avoid specifications are conflicting with a tolerance of μ and is given by: $\theta_i(t) = \theta_o(1 - \mathsf{sign}(\psi_i(t)))$, where $\theta_o \in \mathbb{R}^+$ controls the deviation of the funnel around the circumvent function.

Further, the non-smooth sign function is approximated by the smooth function $\tanh$. When the conflict is resolved, $\theta_i(t)$ becomes 0 and the second part decays $\alpha_i(t)$ exponentially back to zero with a rate of decay governed by constant κ. An example of how the circumvent function modifies the funnel is shown in Figure 1 (c).

Lemma 4.1: $\gamma_s(t), \dot{\gamma}_s(t), \gamma_d(t), \dot{\gamma}_d(t) \in \mathcal{C}$, where $\gamma_L = \mathsf{col}(\gamma_{1,L}, \ldots, \gamma_{n,L})$, $\gamma_U = \mathsf{col}(\gamma_{1,U}, \ldots, \gamma_{n,U})$, $\gamma_d = \mathsf{diag}(\gamma_{1,U} - \gamma_{1,L}, \ldots, \gamma_{n,U} - \gamma_{n,L})$, and $\gamma_s = \gamma_U + \gamma_L$.

Proof: The proof follows a similar approach to that of Lemma 1 in [13]. ■

C. Controller Design

In this section, utilizing the adaptive funnel, discussed in the previous section, we derive the control law to solve Problem 2.1. The controller design is done in three stages.

Stage I. Given an initial state $x(0)$ and target state $\mathbf{T}$, construct the funnel constraints (2) that guide the system trajectory to the target $\mathbf{T}$, as discussed in Section III.

Stage II. Given the unsafe region $\mathbf{U}$, obtain $\mathcal{U}^{\hat{j}}$ as shown in (8) and compute the circumvent function according to (10) or (11). Now modify the funnel around the circumvent function, as discussed in Section IV-B, and determine the adaptive funnel framework, defined by γ_L and γ_U.

Stage III. For the modified funnel, we define the normalized error $\hat{e}(x,t)$, the transformed error $\hat{\varepsilon}(x,t)$ and the diagonal matrix, $\hat{\xi}(x,t)$ as:

$$\hat{e}(x,t) = 2\gamma_d(t)^{-1}\left(x - \frac{1}{2}\gamma_s(t)\right), \tag{15}$$

$$\hat{\varepsilon}(x,t) = \mathsf{col}\left(\ln\left(\frac{1+\hat{e}_1(x_1,t)}{1-\hat{e}_1(x_1,t)}\right),\ldots,\ln\left(\frac{1+\hat{e}_n(x_n,t)}{1-\hat{e}_n(x_n,t)}\right)\right). \tag{16}$$

$$\hat{\xi}(x,t) = 4\gamma_d^{-1}(I_n - \hat{e}(x,t)\hat{e}^\top(x,t))^{-1}. \tag{17}$$

The corresponding constrained region $\hat{\mathbb{D}}$ can be represented by: $\hat{\mathbb{D}} := \{\hat{e}(x,t) : \hat{e}(x,t) \in (-1,1)^n\}$.

Now, in Theorem 4.2, we propose the control strategy $\hat{u}(x,t)$ to satisfy reach-avoid-stay specifications.

Theorem 4.2: Consider a nonlinear control-affine system $\mathcal{S}$ given in (1), assigned a reach-avoid task expressed mathematically through (3) and (10, 11) respectively. If the initial state $x(0)$ is within the modified funnel (Section IV-B), then the control strategy

$$\hat{u}(x,t) = -g(x)^\top(g(x)g(x)^\top)^{-1} \left(\hat{k}\hat{\xi}(x,t)\hat{\varepsilon}(x,t) - \frac{1}{2}\dot{\gamma}_d(t)\hat{e}(x,t)\right). \tag{18}$$

will drive the state trajectory $x(t)$ to the target $\mathbf{T}$ while avoiding the unsafe set $\mathcal{U}^{\hat{j}}$ (8) and adhering to state constraints, i.e., $\exists t \in \mathbb{R}_0^+ : x(t) \in \mathbf{T}$ and $\forall s \in [0,t], x(s) \notin \mathcal{U}^{\hat{j}}$ and $x(s) \in \mathbf{X}$. $\hat{k} \in \mathbb{R}^+$.

Proof: The proof comprises three steps. First, we show that there exists a maximal solution for the normalized error $\hat{e}(x,t)$, which implies that $\hat{e}(x,t)$ remains within $\hat{\mathbb{D}}$ in the maximal time solution interval $[0,\tau_{\max})$. Next, show that the proposed control law (18) constraints $\hat{e}(x,t)$ to a compact subset of $\hat{\mathbb{D}}$. Finally, prove that $\tau_{\max}$ can be extended to ∞.

Step 1. Taking derivatives of (15) and (16), we have:

$$\dot{\hat{e}} = 2\gamma_d^{-1}\left(\dot{x} - \frac{1}{2}\dot{\gamma}_s - \frac{1}{2}\dot{\gamma}_d\hat{e}\right), \text{ and } \dot{\hat{\varepsilon}} = 2(I_n - \hat{e}\hat{e}^\top)^{-1}\dot{\hat{e}}. \tag{19}$$

Substituting the controller (18) in the system dynamics (1), we obtain the closed-loop dynamics:

$$\dot{x} = H_1(x,\hat{e},t) := f(x) + \left(-\hat{k}\hat{\xi}\hat{\varepsilon} - \frac{1}{2}\dot{\gamma}_d\hat{e}\right)$$

and substituting the above equation in $\dot{\hat{e}}$, we obtain

$$\dot{\hat{e}} = H_2(x,\hat{e},t) := 2\gamma_d^{-1}\left(H_1(x,\hat{e},t) - \frac{1}{2}\dot{\gamma}_s - \frac{1}{2}\dot{\gamma}_d\hat{e}\right).$$

Consider the augmented state $y = [x,\hat{e}]^\top$ and its derivative $\dot{y} = H(y,t) := [H_1(x,\hat{e},t), H_2(x,\hat{e},t)]^\top$.

Since the initial state $x(0)$ is within the updated funnel, the initial normalized error $\hat{e}(x(0),0)$ is within the constrained region $\hat{\mathbb{D}}$. Note, that $\hat{\mathbb{D}}$ is an open and bounded set. Further, define $\hat{\mathbb{D}}_x := \{x \in \mathbb{R}^n | \hat{e}(x(0),0) \in \hat{\mathbb{D}}\}$, which is a non-empty open and bounded set. Thus, $\hat{\mathbb{D}}_y := \hat{\mathbb{D}}_x \times \hat{\mathbb{D}}$ is also a non-empty open and bounded set and the initial condition of the augmented state satisfy $y(0) = [x(0), \hat{e}(x(0),0)]^\top$ Therefore, we have the following initial value problem at hand: $\dot{y} = H(y,t), y(0) \in \hat{\mathbb{D}}_y$.

We can see that $\hat{\varepsilon}$ (16), $\hat{\xi}$ (17) and $\dot{\gamma}_d\hat{e}$, defined on $\hat{\mathbb{D}}_y$, are locally Lipschitz continuous in $\hat{e}$. Further, according to Assumption 1, $f(x)$ is also Lipschitz continuous on $\hat{\mathbb{D}}_y$ in x. Therefore, we can conclude that $H(y,t)$ is locally Lipschitz continuous on $\hat{\mathbb{D}}_y$ in y.

Hence, according to Lemma [16, Theorem 54], there exists a maximal solution of the IVP $\dot{y} = H(y,t), y(0) \in \hat{\mathbb{D}}_y$ in the time interval $[0,\tau_{\max})$: $y(t) \in \hat{\mathbb{D}}_y \forall t \in [0,\tau_{\max})$.

Step 2. Based on Step 1, we know, $y(t) \in \hat{\mathbb{D}}_y \implies \hat{e}(t) \in \hat{\mathbb{D}} \implies \gamma_L(t) < x(t) < \gamma_U(t), \forall t \in [0,\tau_{\max})$.

Consider the following positive definite and radially unbounded Lyapunov function candidate: $V = \frac{1}{2}\hat{\varepsilon}^\top\hat{\varepsilon}$.

Differentiating V with respect to time t, substituting $\dot{\hat{e}}$, $\dot{\hat{\varepsilon}}$ (19) and system dynamics $\dot{x}$ (1), and employing the control strategy $u = \hat{u}(x,t)$ (18), we obtain:

$$\begin{aligned}
\dot{V} &= \hat{\varepsilon}^\top\dot{\hat{\varepsilon}} = \hat{\varepsilon}^\top\frac{2}{1-\hat{e}^\top\hat{e}}\dot{\hat{e}} = \hat{\varepsilon}^\top\hat{\xi}\left(\dot{x} - \frac{1}{2}(\dot{\gamma}_s - \dot{\gamma}_d\hat{e})\right) \\
&= \hat{\varepsilon}^\top\hat{\xi}\left(-k\hat{\xi}\hat{\varepsilon} + \left(f(x) - \frac{1}{2}\dot{\gamma}_s\right)\right) \\
&\leq \left\|\hat{\varepsilon}^\top\hat{\xi}\left(-k\hat{\xi}\hat{\varepsilon} + \left(f(x) - \frac{1}{2}\dot{\gamma}_s\right)\right)\right\| \\
&\leq -k\|\hat{\varepsilon}\|^2\|\hat{\xi}\|^2 + \|\hat{\varepsilon}\|\|\hat{\xi}\|\|\hat{\Phi}\|,
\end{aligned}$$

where $\hat{\Phi} := f(x) - \frac{1}{2}\dot{\gamma}_s$. We will look at the boundedness of the two terms in $\hat{\Phi}$ separately. First, $f(x)$ is a continuous function of x and $x \in \hat{\mathbb{D}}_x, \forall t \in [0,\tau_{\max})$, an open and bounded set. Thus, by applying the extreme value theorem, we can infer $\|f(x)\| < \infty$. Secondly, from Lemma 4.1 we know that $\dot{\gamma}_s$ is also bounded. Hence, $\hat{\Phi} \in \mathcal{C}, \forall t \in [0,\tau_{max}]$.

Now add and substract $k\hat{\theta}\left\|\hat{\varepsilon}\right\|^2\left\|\hat{\xi}\right\|^2$, where $0 < \hat{\theta} < 1$

$$\begin{aligned}
\dot{V} &\leq -k(1-\hat{\theta})\left\|\hat{\varepsilon}\right\|^2\left\|\hat{\xi}\right\|^2 - \left\|\hat{\varepsilon}\right\|\left\|\hat{\xi}\right\|\left(k\hat{\theta}\left\|\hat{\varepsilon}\right\|\left\|\hat{\xi}\right\| - \|\hat{\Phi}\|\right) \\
&\leq -k(1-\hat{\theta})\left\|\hat{\varepsilon}\right\|^2\left\|\hat{\xi}\right\|^2, \forall k\hat{\theta}\left\|\hat{\varepsilon}\right\|\left\|\hat{\xi}\right\| - \|\hat{\Phi}\| \geq 0 \\
&\leq -k(1-\hat{\theta})\left\|\hat{\varepsilon}\right\|^2\left\|\hat{\xi}\right\|^2, \forall \left\|\hat{\varepsilon}\right\| \geq \frac{\|\hat{\Phi}\|}{k\hat{\theta}\|\hat{\xi}\|}, \forall t \in [0,\tau_{\max}).
\end{aligned}$$

Therefore, there exists a time-independent upper bound $\hat{\varepsilon}^* \in \mathbb{R}_0^+$ to $\hat{\varepsilon}$, i.e., $\|\hat{\varepsilon}\| \leq \hat{\varepsilon}^*, \forall t \in [0,\tau_{\max})$.

Further, we know from (16) that $\hat{\varepsilon}_i = \ln\left(\frac{1+\hat{e}_i}{1-\hat{e}_i}\right)$. Taking inverse, we can bound $\hat{e}(x,t) = \mathsf{col}(\hat{e}_1,\ldots,\hat{e}_n)$ as:

$$-1 < \frac{\hat{e}_i^{-\hat{\varepsilon}_i^*} - 1}{\hat{e}_i^{-\hat{\varepsilon}_i^*} + 1} =: \hat{e}_{i,L} \leq \hat{e}_i \leq \hat{e}_{i,U} := \frac{\hat{e}_i^{\hat{\varepsilon}_i^*} - 1}{\hat{e}_i^{\hat{\varepsilon}_i^*} + 1} < 1 \quad \forall t \in [0,\tau_{\max}), \text{ for } i \in [1;n].$$

Therefore, by employing the control law (18), we can constrain $\hat{e}$ to a compact subset of $\hat{\mathbb{D}}$ as:

$$\hat{e}(x,t) \in [\hat{e}_L, \hat{e}_U] =: \hat{\mathbb{D}}' \subset \hat{\mathbb{D}}, \forall t \in [0,\tau_{\max}), \tag{20}$$

where, $\hat{e}_L = \mathsf{col}(\hat{e}_{1,L},\ldots,\hat{e}_{n,L}), \hat{e}_U = \mathsf{col}(\hat{e}_{1,U},\ldots,\hat{e}_{n,U})$

Step 3. Finally, we show $\tau_{\max}$ can be extended to ∞.

We know that $\hat{e}(x,t) \in \hat{\mathbb{D}}', \forall t \in [0,\tau_{\max})$, where $\hat{\mathbb{D}}'$ is a non-empty compact subset of $\hat{\mathbb{D}}$. Consequently, $x(t) =$

$\frac{1}{2}\gamma_d\hat{e} + \gamma_s$ also evolves in a compact set: $x(t) \in \hat{\mathbb{D}}'_x \subset \mathbb{D}_x, \forall t \in [0, \tau_{\max})$. Define the compact set $\hat{\mathbb{D}}'_y := \hat{\mathbb{D}}'_x \times \hat{\mathbb{D}}'$ and note that $\hat{\mathbb{D}}'_y \subset \hat{\mathbb{D}}$. Therefore, there is no $t \in [0, \tau_{\max})$ such that $y(t) \notin \hat{\mathbb{D}}_y$.

However, if $\tau_{\max} < \infty$ then according to Lemma [16, Proposition C.3.6], $\exists t' \in [0, \tau_{\max})$ such that $y(t) \notin \hat{\mathbb{D}}_y$. This leads to a contradiction! Hence, we conclude that $\tau_{\max}$ can be extended to ∞, i.e., $x(t)$ satisfies the funnel constraints in (2) for all time $t \geq 0$. In conclusion, the satisfaction of (2) is guaranteed for all time when we employ the control strategy (18). ■

Remark 4.3: From Assumption 1, we know $g(x)g^\top(x)$ is invertible. (20) entails that $\hat{e}$ is bounded. And by definitions (15) and (17), $\hat{\xi}$ and $\hat{\varepsilon}$ are also bounded. Further, from Lemma 4.1, $\dot{\gamma}_d$ also $\in \mathcal{C}$. Hence, the control law $\hat{u}(x,t)$ (18) is well-defined, i.e., continuous, smooth, and bounded.

Figure 2 provides a 3D visualization of a scenario where $\mathbf{X} \subset \mathbb{R}^2$ and the modified funnel circumvents around the unsafe zone $\mathbf{U}$, providing a safe path for the trajectory to reach the target $\mathbf{T} \subset \mathbf{X}$ while staying clear of $\mathbf{U}$.

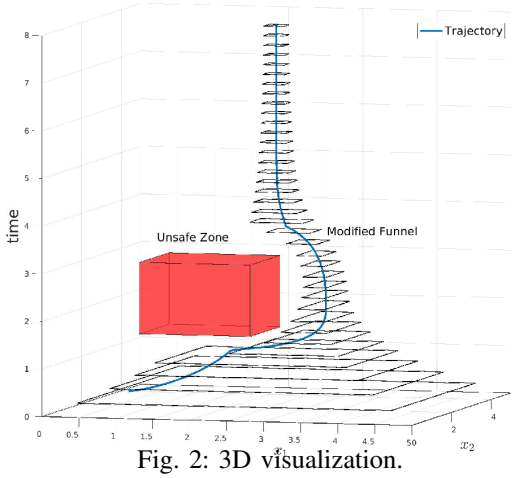


Fig. 2: 3D visualization.

V. Extension to Tackle General Unsafe Sets

Given the unsafe region $\mathbf{U}$ with n_u compact and convex sets, we first choose $\mathcal{U}^{\hat{j}}$ (8). The control law $\hat{u}(x,t)$ (18) ensures that the controlled state trajectory reaches the target while avoiding this $\mathcal{U}^{\hat{j}}$. Now, to address $\mathcal{U}^j$ for $j = [1; n_u]$, we iterate through this procedure until the controlled system trajectory stays entirely clear of the unsafe region $\mathbf{U}$.

Further, in each iteration, for defining the β function, we have a certain degree of randomness. We randomly select $i^{\hat{j}}$ from all the possible alternatives (9). Moreover, we also randomly choose whether to introduce $\beta_i^{\hat{j}}$ in the lower constraint (10) or upper constraint (11), as discussed in Section IV-A. This randomness allows exploration of all the paths around unsafe regions, thus resulting in a higher probability of obtaining a closed-form controller satisfying reach-avoid-stay specifications for complex environments. The algorithm is presented in Algorithm 1.

Algorithm 1 Extension for general unsafe region

Input: $\mathbf{X}, \mathbf{T}, \mathbf{U} = \{\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{n_u}\}, x(0)$
Output: $\hat{u}(x(0), \mathbf{X}, \mathbf{T}, \mathbf{U}, x, t) : \{\exists \tau \in \mathbb{R}_0^+ : x(\tau) \in \mathbf{T}$ and $\forall t \in \mathbb{R}_0^+ : x(t) \in \mathbf{X}, x(t) \cap \mathbf{U} = \emptyset\}$

1. Given $\mathbf{T}$, choose $\eta \in int(\mathbf{T})$ and construct funnel constraints to enforce reachability (2)
2. Apply control law $u(x,t)$ (7) to drive the controlled trajectory $x(t)$ to the target while remaining within the state limits.
3. **while** true **do**
4. Obtain obstacle $\mathcal{U}_{\hat{j}} \in \mathbf{U}$ (8) and introduce the circumvent function β (10) or (11) to modify the funnel around the obstacle as discussed in Section IV-B.
5. Apply control law $\hat{u}(x,t)$ (18) and obtain the controlled trajectory $x_u(t)$.
6. **if** $(x_u(t) \cap \mathcal{U}_j = \emptyset, \forall j \in [1; n_u])$
7. **return** $\hat{u}(x,t)$
8. **end**
9. **end**

Corollary 5.1: Thus, given a system $\mathcal{S}$ in (1), target set $\mathbf{T}$ in the state space $\mathbf{X}$ and unsafe region $\mathbf{U}$, termination of the Algorithm 1 defines an adaptive funnel framework and provides us a well-defined closed-form control law (18) that will guide the system trajectory to the target while avoiding the unsafe region, enforcing reach-avoid-stay specifications.

VI. Simulation Results

To demonstrate, we use an omnidirectional robot [14] for navigation in a 2D arena. The funnel for guiding the robot towards the target is shaped according to (3) with the following parameters: $\rho_{i,0} = 1, \rho_{i,\infty} = 0.05$ and $l_i = 0.7$ for $i \in 1, 2$. $i = 1$ and $i = 2$ represents the x_1 and x_2 coordinates respectively. For the circumvent function (10) or (11), we define $k = 0.001$, $\delta B = 0$ and $\delta t = 0.1$ for all the obstacles. Finally the adaptive law (14) is established with $\mu = 10$, $\kappa = 0.3$ and $\theta_o = 0.1$. The simulation results with three different initial states are depicted in Figure 3.

Path planning and following algorithms offer solutions but lack formal quality guarantees, while symbolic control techniques ensure guarantees with increased computational complexity. We tested the proposed approach against other methods in 500 different arenas, using MATLAB on a computer with an Intel Core i7-11700 Processor and 16 GB RAM. Table I highlights the effectiveness of the proposed approach.

VII. Conclusion and Future Work

In this work, we consider the controller synthesis problem for reach-avoid-stay specification. Given state space constraints, obstacles, and targets, we first proposed the introduction of a circumvent function and the construction of an adaptive funnel framework. We have then derived a closed-form control law ensuring that the trajectories of a nonlinear

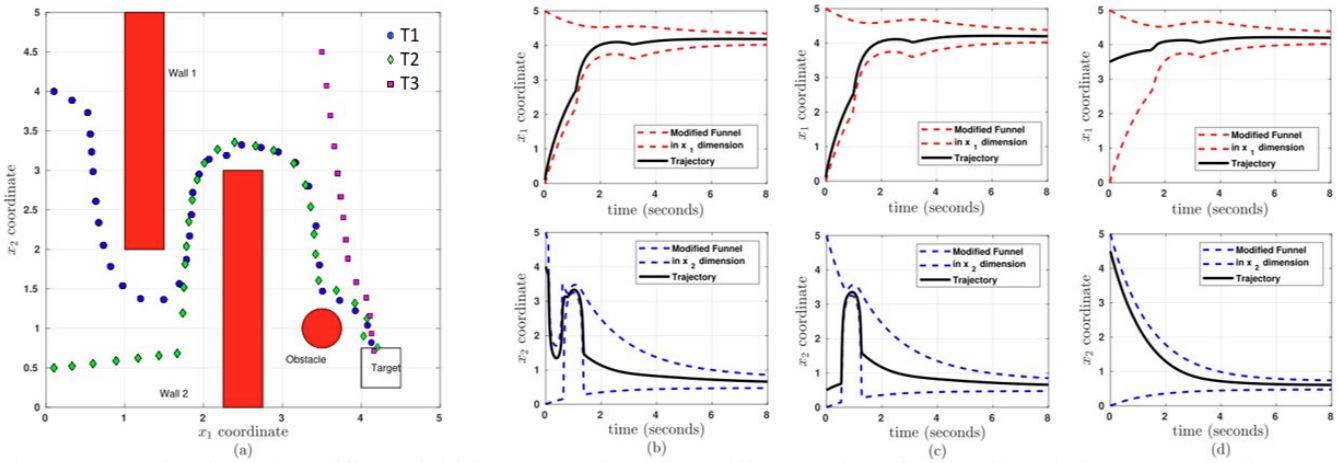


Fig. 3: (a) Starting from three different initial states, we have three different trajectories: T1 (blue circle), T2 (green diamond), and T3 (magenta square). Adaptive funnel framework with controlled system trajectories for (b) T1, (c) T2, and (d) T3.

TABLE I: Comparing the proposed approach with classical algorithms

Algorithm	**Success Rate (%)**	**Computation Time (secs)**	**Formal Guarantees**
A* [17] [18]	95.40	3.32	Optimality under certain conditions } For tracking, we need additional mechanisms
RRT* [17] [18]	97.40	10.57	Probabilistic completeness } such as PID, MPC, Funnel-based control, etc.
RL [19]	99.20	99.31	No formal guarantee
Symbolic Control [20]	100.00	295.46	Formal guarantees under accurate modelling
Adaptive Funnel Control	100.00	5.14	Formal guarantees with partially unknown dynamics

system reach the target while avoiding all the unsafe regions and respecting state-space constraints, thus, enforcing reach-avoid-stay specifications. Finally, the efficacy of the proposed approach is demonstrated through simulation studies.

While the current approach is restricted to fully actuated systems, as considered in Assumption 1, in future work we will extend it to more general underactuated control. Moreover, the efficacy of the approach to address reach-avoid specifications will further be extended to tackle general specifications expressed through regular expressions.

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