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Nonlinear control of inertia wheel pendulum via exact feedback linearization*

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***Abstract*— This paper presents the development and implementation of a real-time state-to-state feedback linearization controller for an inertia wheel pendulum. Based on the nonlinear model of the plant, a mathematical proof of existence of such controller is first given, then a non-trivial local diffeomorphism is found that puts the system in the normal form, and finally the latter is employed to design a nonlinear control law that performs state-to-state exact feedback linearization to maintain the inertia wheel pendulum in the vertical upward position. Details guaranteeing reproducibility of results are provided, both for simulation and real-time implementation.**

I. Introduction

Underactuated nonlinear plants are often used for research purposes in order to demonstrate the effectiveness of control schemes because they are challenging in comparison with fully actuated systems and can be easily related to well-known applications. Typical underactuated nonlinear systems include plants such as the double pendulum also known as pendubot [1], the rotatory pendulum also known as the Furuta one [2], the inverted pendulum on a cart [3], and the reaction wheel pendulum [4] this work is concerned about.

The reaction wheel pendulum, also know as the inertia wheel pendulum, has been used for research purposes in many works testing control law methodologies, for instance, based on sliding modes [5]–[7], computed-torque with prescribed limits [8], [9], model-free control techniques [10], active disturbance rejection [11], robust control handling parametric uncertainties and external disturbances [12], and many more [13], [14].

Among nonlinear control schemes, feedback linearization is inspired in an easy-to-understand idea: cancel out the nonlinearities of the system via a proper feedback with a virtual input; then stabilize the resulting linear system by any known linear technique, e.g., pole placement [15], [16]. As expected, not any system is amenable to this scheme: its implementation requires the system to be structured as a chain of integrators also known as the normal form [17]. Many methodologies rely on this specific form to be applied, e.g., super-twisting sliding modes [18], adaptive control for systems subject to disturbance and saturated control signal [19], methodologies using Lyapunov functions based on homogeneous controllers [20], optimal control with constraints [21], and many others.

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There are different ways to reach a form that consists in a chain of integrators; all of them require finding an appropriate transformation for the nonlinear system, e.g., the standard linear representation for controllable linearizable systems (known as the Brunovsky normal form) can be obtained by a linear transformation [22]. In general, conditions for a nonlinear system to be amenable to the normal form require a series of concepts from the differential geometry field, namely Lie algebra; they have been expressed in the form of sufficient and necessary conditions in [23], with a variety of specific formulae for feedback transformations already available in the literature by the time [24], [25].

Applying control schemes via feedback linearization to the inertia wheel pendulum is not novel; we can find some solutions in [14] and [26], which design control laws based on feedback linearization, but the first one uses a simplified 3rd-order model while the second one does not delve with enough detail to guarantee reproducibility of results. In this work the existence of a diffeomorphism capable of transforming the nonlinear model into the normal form is first established via Lie bracket conditions and involutive distributions; then, the mapping is deduced by constructive means solving a chain of partial differential equations; finally, the transformation is put at use to obtain a normal form, based on which a state-to-state feedback linearization control law is employed along with pole placement for the virtual input of the resulting linear system.

This paper is organized as follows: Section II provides the basis required to transform a nonlinear system into the normal form for applying a control law based on exact linearization; it also describes the inertia wheel pendulum plant this work is aimed to; in Section III, the procedure to find a suitable transformation and the design of the control law by exact linearization are fully detailed. Section IV shows the results of applying the designed control law both in simulation and real-time implementation. Finally, Section V presents the conclusions.

II. Preliminaries

This section has two parts: the first one is concerned with the normal form and conditions for feedback linearization; the second one describes the inertia wheel pendulum by Quanser [27], on which feedback linearization is performed.

A. Feedback linearization and the normal form

Consider the nonlinear system

$$\dot{x}(t) = f(x) + g(x)u(t),\ y(t) = h(x), \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}^p$ is the output; $f(x)$, $g(x)$, and $h(x)$ are nonlinear mapping of appropriate dimensions where $f(0) = 0$. The following theorem gives sufficient and necessary conditions for a nonlinear system (1) to be feedback linearizable:

Theorem 1: [28] The system (1) is feedback linearizable if and only if there is a domain $\mathcal{D} \subset \mathbb{R}^n$, $0 \in \mathcal{D}$ such that

1) the matrix $[g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)]$ has rank n for all $x \in \mathcal{D}$;
2) the distribution span$\{g, ad_f g, \ldots, ad_f^{n-2}\}$ is involutive in $\mathcal{D}$,

where the notation ad_f^k denotes nested Lie brackets such that $ad_f^0 g(x) = g(x)$, $ad_f g(x) = [f, g](x)$, and $ad_f^k g(x) = [f, ad_f^{k-1} g](x)$ with

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x),$$

Proof: See [28, Appendix C.22]. ∎

Control laws based on feedback linearization require transforming the system using a diffeomorphism (at least locally invertible, possibly nonlinear, coordinate change), enabling grouping the inputs in the last set of state equations. There are many methods to do this, with the normal form being the conventional result. The concept of relative degree is crucial to this end: a system (1) has relative degree r if its output has to be differentiated r times for the input u to appear explicitly. With this at hand, the following transformation can be defined [28]:

$$\begin{bmatrix} \eta \\ -- \\ \xi \end{bmatrix} = \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-r}(x) \\ ---- \\ h(x) \\ \vdots \\ L_f^{r-1} h(x) \end{bmatrix} \equiv T(x), \qquad (2)$$

where $\xi \in R^r$, $\eta \in R^{n-r}$, and functions $\phi_1(\cdot)$, $\phi_2(\cdot)$, …, $\phi_{n-r}(\cdot)$ are chosen such that the following holds for $x \in \mathcal{D} \subset \mathbb{R}^n$, $0 \in \mathcal{D}$:

1) $T(0) = 0$ with $T(x)$ is invertible.
2) $\dfrac{\partial \phi_i}{\partial x} g(x) = 0$, $1 \le i \le n - r$.

It can be checked that applying the transformation (2) to the system (1) leads to the desired normal form:

$$\begin{aligned} \begin{bmatrix} \dot{\eta} \\ \dot{\xi} \end{bmatrix} &= \begin{bmatrix} f_0(\eta, \xi) \\ A_c \xi + B_c \gamma(x) \left(u - \alpha(x)\right) \end{bmatrix} \\ y &= C_c \xi, \end{aligned}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{r \times r}, B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{r \times 1}, C_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}_{r \times 1}^T,$$

$$f_0(\eta, \xi) = \left.\frac{\partial \phi}{\partial x} f(x)\right|_{x = T^{-1}(\eta, \xi)}, \ \gamma(x) = L_g L_f^{r-1} h(x),$$
$$\alpha(x) = -\frac{L_f^r h(x)}{L_g L_f^{r-1} h(x)},$$

with $L_f(\cdot)$ and $L_g(\cdot)$ denoting Lie derivatives of their arguments with respect to the vector fields $f(\cdot)$ and $g(\cdot)$, respectively, i.e.:

$$L_f^0 h(x) \equiv h(x), \ L_f h(x) \equiv \frac{\partial h}{\partial x} f(x),$$
$$L_f^k h(x) \equiv L_f L_f^{k-1} h(x), \ L_g L_f h(x) = \frac{\partial (L_f h(x))}{\partial x} g(x).$$

If the relative degree equals the system order, transformation (2) adopts a particular form that allows state-to-state feedback linearization:

Theorem 2: If (1) has relative degree $r = n$, there exists a neighborhood $\mathcal{N}$ for every initial conditions $x_0 \in \mathcal{D}$ such that the map

$$T(x) = \begin{bmatrix} h(x) & L_f h(x) & \ldots & L_f^{n-1} h(x) \end{bmatrix}^T \qquad (3)$$

is a diffeomorphism on $\mathcal{N}$.

Proof: See [28, Appendix C.21]. ∎

The result of applying transformation (3) is a system of the form

$$\dot{\xi} = A_c \xi + B_c \gamma(x) \left(u - \alpha(x)\right), \qquad (4)$$

with A_c, B_c, $\gamma(x)$, and $\alpha(x)$ defined as before with $r = n$.

Based on (4), the control law

$$u(t) = \gamma^{-1}(x) v + \alpha(x), \qquad (5)$$

with v as a virtual input, cancels out the nonlinearities of the system leading to

$$\dot{\xi} = A_c \xi + B_c v,$$

which can be stabilized by any linear technique since the pair (A_c, B_c) is controllable.

B. The inertia wheel pendulum

The mechatronics control kit of Quanser [27] is a set of underactuated plants used for academic and research purposes; one of these plants is the inertia wheel pendulum which is shown in Figure 1. This system consists of a freely rotating arm (vertical plane) attached to a fixed frame, with a DC motor situated at the opposite end to rotate a wheel. The mathematical model of the reaction wheel pendulum, with the origin $x = 0$ positioned vertically upwards, is $\dot{x} = f(x) + g(x)u$ with

$$f(x) = \begin{bmatrix} x_2 \\ \dfrac{gc \sin(x_1 + \pi)}{a} \\ x_4 \\ -\dfrac{gc \sin(x_1 + \pi)}{a} \end{bmatrix}, \ g(x) = \begin{bmatrix} 0 \\ -\dfrac{1}{a} \\ 0 \\ \dfrac{a + b}{ab} \end{bmatrix}, \qquad (6)$$

where x_1 and x_3 are the arm and wheel angles, respectively, x_2 and x_4 their time derivatives, and

$$a = \frac{m_{l1} l_{c1}^2 + l_r^2 m_{l2} + I_{l1}}{\tau}, b = \frac{I_{l2}}{\tau}, c = \frac{m_{l1} l_{c1} + l_r m_{l2}}{\tau},$$

with $m_{l1} = 0.2164$ kg, $l_r = 0.127$ m, $l_{c1} = 0.1173$ m, $m_{l2} = 0.085$ kg, $I_{l1} = 2.225\times10^{-4}$ kg·m^2, $I_{l2} = 24.9526\times 10^{-6}$ kg·m^2, $\tau = 0.0049443$ N·m/V, and $g = 9.804$ m/s^2.

III. Main Results

This section checks that the system under consideration is feedback linearizable, finds a transformation to its normal form, and finds a control law by feedback linearization.

A. Checking if the system is feedback linearizable

Using Theorem 1 with f and g defined in (6) we have that based on

$$ad_f^0 g(x) = g(x) = \begin{bmatrix} 0 & -\frac{1}{a} & 0 & \frac{a+b}{ab} \end{bmatrix}^T,$$

$$ad_f^1 g(x) = [f,g](x) = \frac{dg}{dx}f - \frac{df}{dx}g = \begin{bmatrix} \frac{1}{a} & 0 & -\frac{1}{a}-\frac{1}{b} & 0 \end{bmatrix}^T,$$

$$ad_f^2 g(x) = [f, ad_f^1 g](x) = \frac{dad_f^1 g(x)}{dx}f - \frac{df}{dx}ad_f^1 g(x)$$
$$= \begin{bmatrix} 0 & \frac{cg}{a^2}\cos x_1 & 0 & -\frac{cg}{a^2}\cos x_1 \end{bmatrix}^T,$$

$$ad_f^3 g(x) = [f, ad_f^2 g](x) = \frac{dad_f^2 g(x)}{dx}f - \frac{df}{dx}ad_f^2 g(x)$$
$$= \begin{bmatrix} -\frac{cg}{a^2}\cos x_1 & -\frac{cg}{a^2}x_2\sin x_1 & \frac{cg}{a^2}\cos x_1 & \frac{cg}{a^2}x_2\sin x_1 \end{bmatrix}^T,$$

the matrix

$$\begin{bmatrix} g(x) & ad_f g(x) & ad_f^2 g(x) & ad_f^3 g(x) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1.08 & 0 & 83.93\cos x_1 \\ -1.08 & 0 & -83.93\cos x_1 & 83.93 x_2 \sin x_1 \\ 0 & -201.08 & 0 & -83.93\cos x_1 \\ 201.08 & 0 & 83.93\cos x_1 & -83.93 x_2 \sin x_1 \end{bmatrix},$$

can be calculated, which is full rank around $x = 0$ as implied by continuity of its rank at $x = 0$

$$\text{rank}\begin{bmatrix} 0 & 1.0817 & 0 & 83.9345 \\ -1.0817 & 0 & -83.9345 & 0 \\ 0 & -201.0817 & 0 & -83.9345 \\ 201.0817 & 0 & 83.9345 & 0 \end{bmatrix} = 4.$$

This verifies the first condition in Theorem 1; the second one concerning an involutive distribution can be checked similarly.

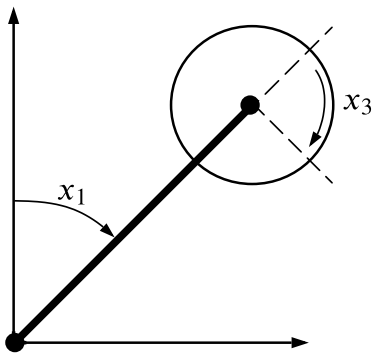


Fig. 1. The inertia wheel pendulum

B. The map for exact feedback linearization

The preceding procedure ensures the existence of a mapping, according to Theorem 2, capable of expressing the inertia wheel pendulum system as $\dot{\xi} = A_c\xi + B_c\gamma(x)(u - \alpha(x))$, which is already state-to-state feedback linearizable. That mapping requires finding a function $h(x)$ such that:

$$\begin{aligned} &L_g L_f^0 h(x) = 0,\ L_g L_f^1 h(x) = 0, \\ &L_g L_f^2 h(x) = 0,\ L_g L_f^3 h(x) \neq 0, \end{aligned} \tag{7}$$

whose existence is guaranteed by the previous section.

The first condition in (7) can be developed as follows:

$$L_g L_f^0 h(x) = 0 \iff L_g h(x) = 0$$
$$\iff \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} & \frac{\partial h}{\partial x_4} \end{bmatrix}\begin{bmatrix} 0 & -\frac{1}{a} & 0 & \frac{a+b}{ab} \end{bmatrix}^T = 0$$
$$\iff \frac{\partial h}{\partial x_2}\left(-\frac{1}{a}\right) + \frac{\partial h}{\partial x_4}\left(\frac{a+b}{ab}\right) = 0,$$

which can be satisfied making $h(x)$ independent of x_2 and x_4. Thus, $h(x) \equiv h(x_1, x_3)$ will be considered in the sequel.

The second condition in (7) can be developed as follows:

$$L_g L_f^1 h(x) = 0$$
$$\iff L_g\left(\begin{bmatrix} \frac{\partial h}{\partial x_1} & 0 & \frac{\partial h}{\partial x_3} & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ \frac{gc\sin(x_1+\pi)}{a} \\ x_4 \\ -\frac{gc\sin(x_1+\pi)}{a} \end{bmatrix}\right) = 0$$
$$\iff L_g\left(x_2\frac{\partial h}{\partial x_1} + x_4\frac{\partial h}{\partial x_3}\right) = 0,$$

where the calculations can be simplified if we assign the structure $h(x_1, x_3) = kx_1 + lx_3$ with constants k y l to be found. Then:

$$L_g\left(x_2\frac{\partial h}{\partial x_1} + x_4\frac{\partial h}{\partial x_3}\right) = 0 \iff L_g\left(kx_2 + lx_4\right) = 0$$
$$\iff \begin{bmatrix} 0 & k & 0 & l \end{bmatrix}\begin{bmatrix} 0 \\ -\frac{1}{a} \\ 0 \\ \frac{a+b}{ab} \end{bmatrix} = 0 \iff -\frac{k}{a} + \frac{l(a+b)}{ab} = 0,$$

from which the following relationships can be deduced

$$l = \frac{bk}{a+b} \ \Rightarrow\ k - l = \frac{ak}{a+b}, \tag{8}$$

where the first one allows replacing any l by an expression depending on k and the second one will be found in several expressions in the sequel.

The third condition in (7) can be developed as follows:

$$
\begin{aligned}
&L_g L_f^2 h(x) = 0 \\
&\iff L_g \left(\begin{bmatrix} 0 & k & 0 & l \end{bmatrix} \begin{bmatrix} x_2 \\ \dfrac{gc\sin(x_1+\pi)}{a} \\ x_4 \\ -\dfrac{gc\sin(x_1+\pi)}{a} \end{bmatrix} \right) = 0 \\
&\iff L_g \left(\frac{kgc\sin(x_1+\pi)}{a} - \frac{lgc\sin(x_1+\pi)}{a} \right) = 0 \\
&\iff \begin{bmatrix} \dfrac{kgc\cos(x_1+\pi)}{a} - \dfrac{lgc\cos(x_1+\pi)}{a} & 0 & 0 & 0 \end{bmatrix} \\
&\qquad \times \begin{bmatrix} 0 & -\dfrac{1}{a} & 0 & \dfrac{a+b}{ab} \end{bmatrix}^T = 0,
\end{aligned}
$$

which always holds, regardless of the values of k and l.

For the fourth condition in (7) we have:

$$
\begin{aligned}
&L_g L_f^3 h(x) \neq 0 \\
&\iff L_g \left(\begin{bmatrix} \dfrac{gc(k-l)\cos(x_1+\pi)}{a} & 0 & 0 & 0 \end{bmatrix} \right. \\
&\qquad \left. \times \begin{bmatrix} x_2 & \dfrac{gc\sin(x_1+\pi)}{a} & x_4 & -\dfrac{gc\sin(x_1+\pi)}{a} \end{bmatrix}^T \right) \neq 0 \\
&\iff L_g \left(\frac{gc(k-l)x_2\cos(x_1+\pi)}{a} \right) \neq 0 \\
&\iff \begin{bmatrix} -\dfrac{gc(k-l)x_2\sin(x_1+\pi)}{a} & \dfrac{gc(k-l)\cos(x_1+\pi)}{a} & 0 & 0 \end{bmatrix} \\
&\qquad \times \begin{bmatrix} 0 & -\dfrac{1}{a} & 0 & \dfrac{a+b}{ab} \end{bmatrix}^T \neq 0 \\
&\iff -\frac{gc(k-l)\cos(x_1+\pi)}{a^2} \neq 0 \iff \frac{gc(k-l)\cos x_1}{a^2} \neq 0
\end{aligned}
$$

which is guaranteed in $\mathcal{D} = \{x \in \mathbb{R}^4 : -\pi/2 < x_1 < \pi/2\}$.

Thus, gathering the information above, we have that

$$
\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = T(x) = \begin{bmatrix} kx_1 + \dfrac{bk}{a+b}x_3 \\ kx_2 + \dfrac{bk}{a+b}x_4 \\ -\dfrac{gck}{a+b}\sin x_1 \\ -\dfrac{gck}{a+b}x_2\cos x_1 \end{bmatrix}, \tag{9}
$$

transforms the inertia wheel pendulum equations into the normal form.

Indeed, taking into account that:

$$
\dot{\xi}_1 = k\dot{x}_1 + \frac{bk}{a+b}\dot{x}_3 = kx_2 + \frac{bk}{a+b}x_4 = \xi_2,
$$

$$
\begin{aligned}
\dot{\xi}_2 &= k\dot{x}_2 + \frac{bk}{a+b}\dot{x}_4 \\
&= k\frac{gc\sin(x_1+\pi)-u}{a} \\
&\quad + \frac{bk}{a+b}\frac{-bgc\sin(x_1+\pi)+(a+b)u}{ab} = \xi_3,
\end{aligned}
$$

$$
\dot{\xi}_3 = -\frac{gck}{a+b}\dot{x}_1\cos x_1 = -\frac{gck}{a+b}x_2\cos x_1 = \xi_4,
$$

$$
\begin{aligned}
\dot{\xi}_4 &= -\frac{gck}{a+b}\left(-x_2\dot{x}_1\sin x_1 + \dot{x}_2\cos x_1\right) \\
&= -\frac{gck}{a+b}\left(-x_2x_2\sin x_1 + \frac{gc\sin(x_1+\pi)-u}{a}\cos x_1\right) \\
&= \frac{gck}{a+b}\left(x_2^2\sin x_1 + \frac{gc}{a}\sin x_1\cos x_1\right) + \frac{gck}{a(a+b)}\cos x_1 u.
\end{aligned}
$$

Now the model can be expressed in the normal form:

$$
\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_c} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{B_c} \underbrace{(-83.483\cos x_1)}_{\gamma(x)} \tag{10}
$$

$$
\times \left(u \underbrace{-2994.5\sin(2x_1) + 77.18x_2^2\sin x_1}_{-\alpha(x)} \right), \tag{11}
$$

where the controllable pair (A_c, B_c) and functions $\alpha(x)$ and $\gamma(x)$ are easy to distinguish.

C. Exact feedback linearization control

Based on the normal form (10) of the inertia wheel pendulum, a control law $u(t) = \gamma^{-1}(x)(v(t) - \alpha(x))$ must be found, where $\gamma^{-1}(x) = -0.0120/\cos x_1$, $\alpha(x) = 2994.5\sin(2x_1) - 77.18x_2^2\sin x_1$, and the virtual input v will be calculated later. Note that $\gamma(x)$ is invertible in $\mathcal{D}$ as defined above, which is a characteristic that we expected.

The virtual control input $v(t)$ for the controllable pair (A_c, B_c) can be designed such that the closed-loop linear system is stable. For this case, using Ackermann's formula to assign the poles in the roots of $\alpha(s) = (s+3)(s+4)(s+6)(s+8)$, we have that:

$$
\begin{aligned}
F &= -\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{[B_c\ A_cB_c\ A_c^2B_c\ A_c^3B_c]^{-1}} \alpha\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} -576 & -504 & -158 & -21 \end{bmatrix},
\end{aligned}
$$

which means $v(t) = F\xi$ can be substituted in $u(t)$ to obtain the final control law in terms of x:

$$
\begin{aligned}
u(t) =& 19.4145x_2 + 71.7388\sin x_1 + \frac{6.8996x_1}{\cos x_1} + \frac{6.0372x_2}{\cos x_1} \\
&+ \frac{0.0371x_3}{\cos x_1} + \frac{0.0325x_4}{\cos x_1} + \frac{146.0710\sin x_1}{\cos x_1} \\
&- \frac{0.9245x_2^2\sin x_1}{\cos x_1},
\end{aligned} \tag{12}
$$

provided $-\pi/2 < x_1 < \pi/2$.

IV. Simulation and real-time Implementation

This section presents the results of implementing the feedback linearization control law (12), developed in the previous section. The simulations are shown first, allowing for analysis of the behavior of the mathematical system and the characteristics exhibited by the control signal. This is done to ensure that the control law is suitable for real-time implementation, including verifying that the magnitude of the control signal is appropriate for the actuator (a DC motor which allows input voltages in the range $[-24, 24]$V). Since the control law (12) depends on the original states, it is ready to be applied without further adaptation.

Figure 2 shows the results of implementing the control law in simulation with an initial condition $x(0) = [0.1\ 0\ 0\ 0]^T$; it illustrates how the underactuated angle x_1 reaches the upward vertical position $x = 0$, indicating the effective stabilization of the system at the desired point by the control law. Angular velocities as well as the actuated angle reach the value of 0, as expected. In Figure 3, the time evolution of the control law is depicted, showing that the reached limits fall within the range allowed by the actuator. Importantly, the model equations have been deduced from the simulation files of the provider; the control law has been applied on the latter.

Given the appropriateness of the control law for implementation, as evidenced by the system's behavior in simulation, real-time setups were tried. Figure 4 illustrates the real-time results of implementing control law (12). It shows how the state x_1, representing the position of the underactuated arm, converges from the initial position to the origin, while maintaining small oscillation possibly due to small model mismatches, quality of actuators, computational sampling, etc. Figure 5 depicts the convergence of the states x_2, x_3, and x_4 towards the origin, illustrating how they all reach this point. Figure 6 shows the control signal which is within acceptable values[1].

[1]Videos of the real-time implementations are available at `https://youtube.com/shorts/nbbDfl3zsw4?feature=share`.

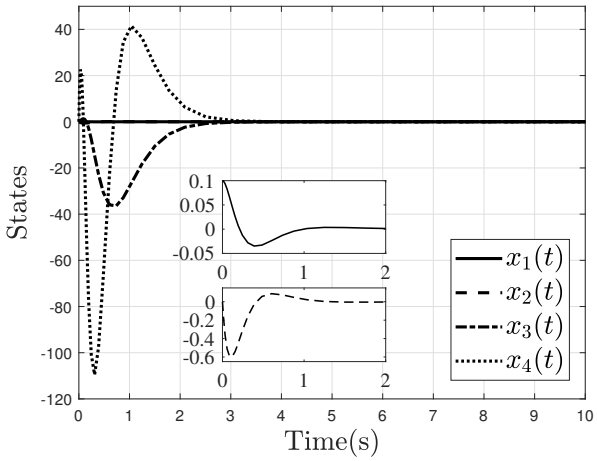


Fig. 2. Evolution of the states for simulation

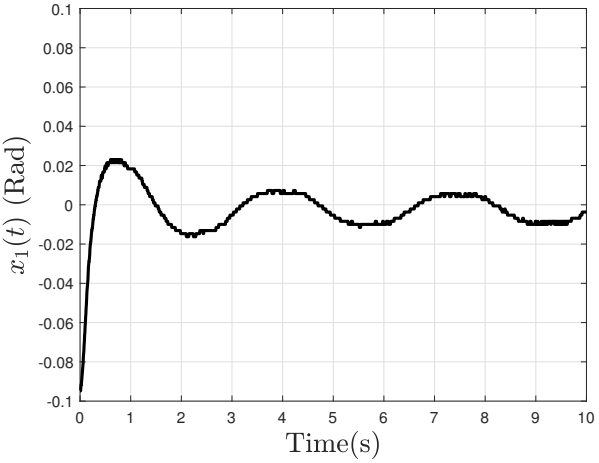


Fig. 4. Evolution of x_1 for real time

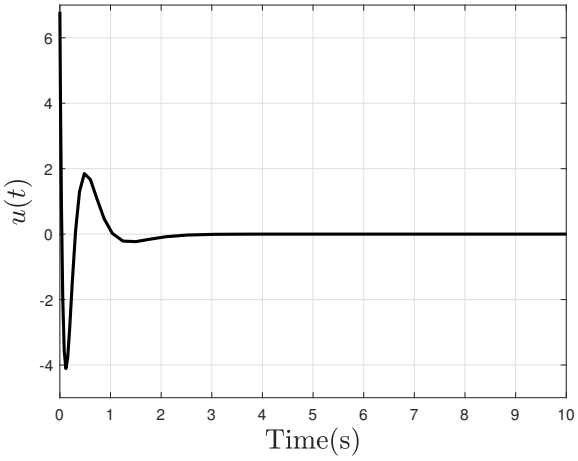


Fig. 3. Evolution of the control signal for simulation

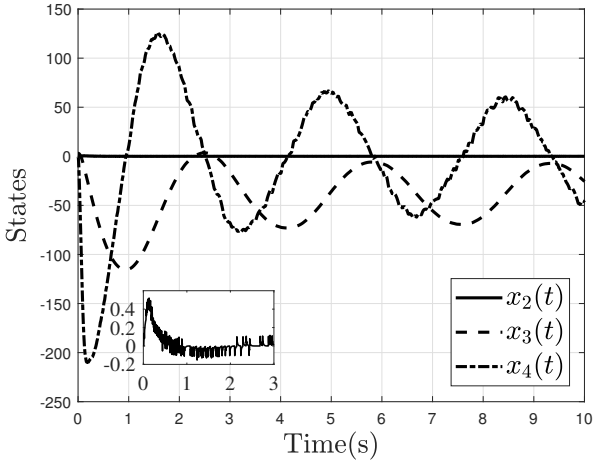


Fig. 5. Evolution of x_2, x_3, and x_4 for real time

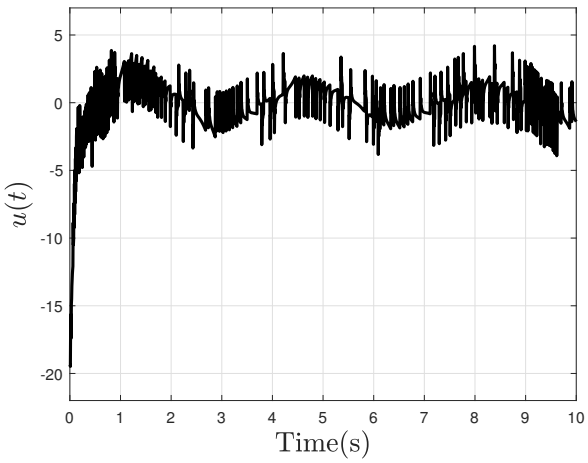


Fig. 6. Evolution of the control signal for real-time

V. Conclusions

A control law based on state-to-state feedback linearization for the inertia wheel pendulum has been presented. The entire procedure for the proper design of the control law has been explicitly demonstrated, considering each requirement to identify the suitable transformation of the system, using the transformation in the original system and thus expressing it in the normal form. The designed control law has been implemented in simulation where its correct operation was verified, then it has been put at test in the inertia wheel pendulum in real time, where stabilization has been achieved.

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